Math 3210 § 2. **Second Midterm Exam** Name: Solutions
Treibergs October 5, 2011

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1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence and $L \in \mathbb{R}$. State the definition: $L = \lim_{n\to\infty} a_n$. Guess the limit. Then use the definition of limit to prove that your guess is correct.* $L = \lim_{n\to\infty} \dfrac{\sqrt{n}}{1+\sqrt{n}}$.

The real sequence $\{a_n\}$ converges to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{R}$ such that $|a_n - L| < \varepsilon$ whenever $n > N$.

We guess $L = 1$. Choose $\varepsilon > 0$. Let $N = \varepsilon^{-2}$. For any $n > N$ we have

$$|a_n - L| = \left|\frac{\sqrt{n}}{1+\sqrt{n}} - 1\right| = \left|\frac{\sqrt{n} - 1 - \sqrt{n}}{1+\sqrt{n}}\right| = \frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{1/\varepsilon^2}} = \varepsilon.$$

2. *Let $E \subset \mathbb{R}$ be a nonempty subset and $m \in \mathbb{R}$. State the definition: $m = \inf E$. (m is also called the greatest lower bound of E.) Consider the set $E = \left\{\dfrac{p}{q} : p \in \mathbb{N} \text{ and } q \in \mathbb{N}\right\}$. Find* $\inf E$ *and prove that it is the infimum.*

m is the greatest lower bound of a nonempty set E if (1) it is a lower bound, namely $(\forall x \in E)(m \le x)$ and (2) it is the greatest of lower bounds, namely $(\forall\varepsilon > 0)(\exists x \in E)(x < m+\varepsilon)$.

We show that $0 = \inf E$ where $E = \{p/q : p, q \in \mathbb{N}\}$. To see that 0 is a lower bound choose $x \in E$. Hence $x = p/q$ for some $p, q \in \mathbb{N}$. But $p > 0$ and $q > 0$, hence $q^{-1} > 0$ which implies $x = pq^{-1} > 0$. Thus we have shown $0 \le x$ for all $x \in E$.

To see that 0 is the greatest of lower bounds choose $\varepsilon > 0$. Hence $\varepsilon^{-1} > 0$. By the Archimedean Property, there is an $q \in \mathbb{N}$ so that $q > \varepsilon^{-1}$, hence $1/q < \varepsilon$. Let $p = 1$. Then there is $x \in E$, namely $x = p/q = 1/q < \varepsilon$. Thus we have shown for any $\varepsilon > 0$ there is $x = p/q \in E$ such that $x < 0 + \varepsilon$ so no number greater than zero is an upper bound.

3. *Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.*

a. Statement: *If $x, y \in \mathbb{R}$ are such that $x \ne y$ and $y \ne 0$ then* $\left|\dfrac{y+x}{y-x}\right| \le \dfrac{|y|+|x|}{|y|}$.

False. Let $x = 1$, $y = 2$ so $\left|\dfrac{y+x}{y-x}\right| = \left|\dfrac{2+1}{2-1}\right| = 3$ which exceeds $\dfrac{|y|+|x|}{|y|} = \dfrac{|2|+|1|}{|2|} = 1.5$.

b. Statement: *Let $\{a_n\}$ be real, convergent sequences such that a_n is irrational for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} a_n$ is irrational.*

False. Let $a_n = \dfrac{\sqrt{2}}{n}$. Then a_n is irrational as it is the product of an irrational $\sqrt{2}$ and a nonzero rational $1/n$. However $a_n \to 0$ as $n \to \infty$ but 0 is rational.

c. Statement: *If $f : \mathbb{R} \to \mathbb{R}$ be a bounded function, then $\inf_{\mathbb{R}} f \le \sup_{\mathbb{R}} f$.*

True. Let $E = \{f(x) : x \in \mathbb{R}\}$. Then $f(0) \in E$. Since $\inf_{\mathbb{R}} f = \inf E$ is a lower bound for E we have $\inf_{\mathbb{R}} f \le f(0)$. Since $\sup_{\mathbb{R}} f = \sup E$ is an upper bound for E we have $f(0) \le \sup_{\mathbb{R}} f$. Thus $\inf_{\mathbb{R}} f \le f(0) \le \sup_{\mathbb{R}} f$.

The Axioms for a Field F with binary operations $+$ and $\cdot$.

A1. $x + y = y + x$ for all $x, y \in F$.;

A2. $x + (y + z) = (x + y) + z$ for all $x, y, z \in F$;

A3. There is an element $0 \in F$ such that $0 + x = x$ for all $x \in F$;

A4. For every $x \in F$ there is an element $-x$ such that $x + (-x) = 0$;

M1. $xy = yx$ for all $x, y \in F$.;

M2. $x(yz) = (xy)z$ for all $x, y, z \in F$;

M3. There is an element $1 \in F$ such that $1 \neq 0$ and $1x = x$ for all $x \in F$;

M4. For each non-zero $x \in F$ there is an element x^{-1} such that $x^{-1}x = 1$;

D. $x(y + z) = xy + xz$ for all $x, y, z \in F$.

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4. *Let $x, y \neq 0$ be elements of the field F. Hence also $xy \neq 0$. Using just the axioms of a field, show that $(xy)^{-1} = x^{-1}y^{-1}$.*

$x^{-1}y^{-1} = 1(x^{-1}y^{-1})$	M3. Multiplicative Identity
$= [(xy)^{-1}(xy)](x^{-1}y^{-1})$	M4. Multiplicative Inverse
$= (xy)^{-1}[(xy)(x^{-1}y^{-1})]$	M2. Associativity of Multiplication
$= (xy)^{-1}[(yx)(x^{-1}y^{-1})]$	M1. Commutativity of Multiplication
$= (xy)^{-1}[((yx)x^{-1})y^{-1}]$	M2. Associativity of Multiplication
$= (xy)^{-1}[(y(xx^{-1}))y^{-1}]$	M2. Associativity of Multiplication
$= (xy)^{-1}[(y(x^{-1}x))y^{-1}]$	M1. Commutativity of Multiplication
$= (xy)^{-1}[(y1)y^{-1}]$	M4. Multiplicative Inverse
$= (xy)^{-1}[(1y)y^{-1}]$	M1. Commutativity of Multiplication
$= (xy)^{-1}[yy^{-1}]$	M3. Multiplicative Identity
$= (xy)^{-1}[y^{-1}y]$	M1. Commutativity of Multiplication
$= (xy)^{-1}1$	M4. Multiplicative Inverse
$= 1(xy)^{-1}$	M1. Commutativity of Multiplication
$= (xy)^{-1}$	M3. Multiplicative Identity

5. *Let $E \subset \mathbb{R}$ be a nonempty subset. State the definition: E is not bounded above. Consider the set $E = \left\{ \dfrac{n^3}{n^2+2} : n \in \mathbb{N} \right\}$. Show that E is not bounded above.*

The nonempty real set E is not bounded above if for every $M \in \mathbb{R}$ there is an $x \in E$ such that $M < x$, *i.e.*, $(\forall M \in \mathbb{R})(\exists x \in E)(M < x)$.

To show that $E = \left\{ \dfrac{n^3}{n^2+2} : n \in \mathbb{N} \right\}$ is not bounded above, choose $M \in \mathbb{R}$. By the Archimedean Property, there is an $n \in \mathbb{N}$ such that $n > M + 2$. For this n, because $n \geq 1$ we have $n^2 \geq n$ so

$$\frac{n^3}{n^2+2} = \frac{n^3 + 2n - 2n}{n^2+2} = n - \frac{2n}{n^2+2} \geq n - \frac{2n^2}{n^2+2} \geq n - \frac{2n^2}{n^2} = n - 2 > (M+2) - 2 = M.$$

Thus we have shown for any $M \in \mathbb{R}$ there is $x \in E$, namely $x = \dfrac{n^3}{n^2+2}$ for this n such that $M < x$.