1. Assume $a \neq 0$. Prove that for every $n \in \mathbb{N}$,

$$
\sum_{k=1}^{n} \frac{a - 1}{a^k} = 1 - \frac{1}{a^n}.
$$

We prove the formula by induction.

**Base Case.** For $n = 1$, the left side of formula (1) is the first term $\sum_{k=1}^{1} \frac{a - 1}{a^k} = \frac{a - 1}{a}$. The right side is $1 - \frac{1}{a} = \frac{a - 1}{a}$. They are equal so the base case is verified.

**Induction Case.** Assume that the formula (1) holds for some $n \in \mathbb{N}$. For $n + 1$

$$
\sum_{k=1}^{n+1} \frac{a - 1}{a^k} = \frac{a - 1}{a^{n+1}} + \sum_{k=1}^{n} \frac{a - 1}{a^k} = \frac{a - 1}{a^{n+1}} + 1 - \frac{1}{a^n}
$$

Using the induction hypothesis (1)

$$
= 1 + \frac{a - 1}{a^{n+1}} - \frac{a}{a^{n+1}} = 1 - \frac{1}{a^{n+1}}.
$$

We conclude that the formula holds for $n + 1$ as well.

Since we have established both the base case and induction case, by mathematical induction, (1) holds for all $n \in \mathbb{N}$.

2. Recall the definition given in class.

Suppose that we have two nonempty sets $A$ and $B$ and a function $f : A \to B$. A function $g : B \to A$ is called an inverse function of $f$ iff

(1.) $f(g(y)) = y$ for all $y \in B$;

(2.) $g(f(x)) = x$ for all $x \in A$;

Let $f : A \to B$ be a function and $E \subset B$ a set. Define $f^{-1}(E)$. Suppose that $f : A \to B$ has an inverse function called $g : B \to A$. Let $E \subset B$. Show that $f^{-1}(E) = g(E)$.

The preimage set is defined to be

$$
f^{-1}(E) = \{x \in A : f(x) \in E\}.
$$

To show $f^{-1}(E) = g(E)$, we first show $f^{-1}(E) \subset g(E)$ and then we show $f^{-1}(E) \supset g(e)$.

To show $f^{-1}(E) \subset g(E)$, we choose an $x \in f^{-1}(E)$ to show that it is in $g(E)$. But by definition of preimage, this means that $f(x) \in E$. Call it $y = f(x) \in E$. Applying $g$, we have $g(y) \in g(E)$ by the definition of image of $g$. But by property (2.) of inverse functions, $x = g(f(x)) = g(y)$. Hence $x \in g(E)$ as to be shown.

To show $f^{-1}(E) \supset g(E)$, we choose an $x \in g(E)$ to show that it is in $f^{-1}(E)$. But by definition of image set, this means that there is a $y \in E$ so that $x = g(y)$. Applying $f$, this means by property (1.) that $y = f(g(y)) = f(x)$. But since $f(x) = y \in E$, this implies by the definition of preimage, that $x \in f^{-1}(E)$, as to be shown.
3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

**Statement 1.** If \( f : A \to B \) and \( E \subseteq F \subseteq A \) are subsets then \( f(E) \subseteq f(F) \).

**TRUE.** Choose \( y \in f(E) \) to show \( y \in f(F) \). Hence there is \( x \in E \) so that \( f(x) = y \). But \( E \subseteq F \) implies \( x \in F \). Thus \( y = f(x) \in f(F) \), as to be shown.

**Statement 2.** For \( E, F \subseteq X \) any two subsets, \( E \setminus F = \emptyset \) implies \( E = F \).

**FALSE.** Take real subsets \( E = [0, 1] \) and \( F = [0, 2] \). Then \( E \setminus F = \emptyset \) but \( E \neq F \).

**Statement 3.** Suppose \( f : A \to B \) is a function. Suppose for every \( x_1, x_2 \in A \), \( f(x_1) \neq f(x_2) \) implies \( x_1 \neq x_2 \). Then \( f \) is one-to-one.

**FALSE.** Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = x^2 \). The condition is equivalent to its contrapositive “if \( x_1 = x_2 \) then \( f(x_1) = f(x_2) \)” which holds for any function, e.g., for \( f \), but \( f \) is not one-to-one because \( f(-2) = 4 = f(4) \).

4. Let \( A, B, C \) be nonempty sets and \( g : A \to B \) and \( f : B \to C \) be functions. Write the definition: \( f : B \to C \) is onto. Show that if the composite function \( f \circ g : A \to C \) is onto then \( f \) is onto.

Give an example that shows that even if \( f \circ g \) is onto, then \( g \) does not need to be onto.

\( f : B \to C \) is onto means that for every \( z \in C \) there is a \( y \in B \) so that \( f(y) = z \).

To show that \( f : B \to C \) is onto, we choose \( z \in C \). Since we assume that \( f \circ g : A \to C \) is onto, there is an \( x \in A \) so that \( f \circ g(x) = z \). Let \( y = g(x) \in B \). Then \( f(y) = f(g(x)) = f \circ g(x) = z \).

Hence we have found a \( y \in B \) so that \( f(y) = z \). Thus we have shown that \( f \) is onto.

Take \( A = \{0\} \), \( B = \{1, 2\} \) and \( C = \{3\} \). Define \( g : A \to B \) by \( g(0) = 1 \). Define \( f : B \to C \) by \( f(1) = f(2) = 3 \). Then \( g \) is not onto because \( g(A) = \{1\} \neq B \). However \( f \circ g \) is onto because \( f \circ g(A) = f \circ g(\{0\}) \).

5. Let \( E \subseteq \mathbb{R} \) be a set of real numbers. Suppose that the set is given by

\[
E = \left\{ x \in \mathbb{R} : (\forall \sigma < 1) \ (\exists \tau > 0) \ \sigma \leq x < \sigma + \tau \right\}.
\]

Write the set \( E \) in terms of unions and intersections. Find the complement \( E^c \) by negating the expression for \( E \) and writing it so that the negators come after the quantifiers. Express \( E^c \) in terms of intervals and prove your result.

In terms of intersections and unions,

\[
E = \bigcap_{\sigma < 1} \bigcup_{\tau > 0} \left[ \sigma, \sigma + \tau \right) \quad \text{(which equals} \quad \bigcap_{\sigma < 1} \left[ \sigma, \infty \right) = [1, \infty) \big).\]

By negating the quantifiers we see that the complement is

\[
E^c = \left\{ x \in \mathbb{R} : \sim (\forall \sigma < 1) \ (\exists \tau > 0) \ \sigma \leq x < \sigma + \tau \right\}
\]

\[
= \left\{ x \in \mathbb{R} : (\exists \sigma < 1) \ (\forall \tau > 0) \ (x < \sigma \ \text{or} \ \sigma + \tau \leq x) \right\}. \quad (2)
\]

We expect that \( E^c = (-\infty, 1) \). We can check this in several ways, but let us argue with \( E^c \) given by formula (2). We first show “\( \subset \)” and then show “\( \supset \).”

To show that \( E^c \subset (-\infty,1) \) we choose \( x \in E^c \). Then there is \( \sigma_0 < 1 \) such that \( (\forall \tau > 0)(x < \sigma_0 \ or \ \sigma_0 + \tau \leq x) \). It follows that \( x < \sigma_0 \). If this were not the case and \( x \geq \sigma_0 \), by taking \( \tau_0 > 0 \) so large that \( \tau_0 > x - \sigma_0 \), neither \( x < \sigma_0 \) nor \( \sigma_0 + \tau_0 \leq x \) is true so that \( (\forall \tau > 0)(x < \sigma_0 \ or \ \sigma_0 + \tau \leq x) \) is false. Thus \( x < \sigma_0 < 1 \) so \( x \in (-\infty, 1) \) as to be shown.

To show that \( E^c \supset (-\infty,1) \) we choose \( x \in (-\infty,1) \) or \( x < 1 \), and show that \( x \in E^c \). If we pick \( \sigma_0 = (1 + x)/2 \) between \( x \) and 1, then \( x < \sigma_0 \) is true and so \( (\forall \tau > 0)(x < \sigma_0 \ or \ \sigma_0 + \tau \leq x) \) is also true for this \( \sigma_0 \). Thus \( x \in E^c \).