Math 3210 § 1. Third Midterm Exam

Name: Solutions
Treibergs
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[1] Let \( \{x_n\} \subset \mathbb{R} \) be a sequence. State the definition: \( \{x_n\} \) is a Cauchy sequence. Let \( x_n = \frac{n}{n+1} \).
Show that \( \{x_n\} \) is a Cauchy sequence.

Definition. \( \{x_n\} \) is a Cauchy sequence if for every \( \epsilon > 0 \) there is an \( R \in \mathbb{R} \) such that \( |x_n - x_m| < \epsilon \) whenever \( m, n \in \mathbb{N} \) satisfy \( m, n > R \).

Proof. Choose \( \epsilon > 0 \). Let \( R = \frac{1}{\epsilon} \). Suppose \( m, n \in \mathbb{N} \) satisfy \( m, n > R \). One is larger, say, \( m \geq n \). Then since \( 0 \leq m - n \leq m + 1 \),
\[
|x_n - x_m| = \left| \frac{m}{m+1} - \frac{n}{n+1} \right| = \frac{|m(n+1) - n(m+1)|}{(m+1)(n+1)} = \frac{m-n}{m+1} \cdot \frac{1}{n+1} \leq \frac{1}{n+1} < \frac{1}{R} = \epsilon. \]

[2.] Let \( f, f_n : D \to \mathbb{R} \) be functions. State the definition: \( f_n \) converges to \( f \) pointwise on \( D \) as \( n \to \infty \). State the definition: \( f_n \) converges to \( f \) uniformly on \( D \) as \( n \to \infty \). Let \( g_n(x) = \frac{x^2}{n^2 + x^2} \) and \( g(x) = 0 \). Show that \( g_n \to g \) pointwise on \( \mathbb{R} \). Does \( g_n \to g \) uniformly on \( \mathbb{R} \)? Prove your answer.

Definition: \( f_n \to f \) converges pointwise on \( D \) means for every \( x \in D \) we have \( \lim_{n \to \infty} f_n(x) = f(x) \). i.e., for every \( x \in D \) and for every \( \epsilon > 0 \) there is \( R \in \mathbb{R} \) such that \( |f_n(x) - f(x)| < \epsilon \) whenever \( n \in \mathbb{N} \) satisfies \( n > R \).

Definition: \( f_n \to f \) converges uniformly on \( D \) means for every \( \epsilon > 0 \) there is \( R \in \mathbb{R} \) such that \( |f_n(x) - f(x)| < \epsilon \) whenever \( x \in D \) and \( n \in \mathbb{N} \) satisfies \( n > R \).

To see that \( g_n \to g \) pointwise, by the workhorse theorem for sequences, for any \( x \in \mathbb{R}, \)
\[
\lim_{n \to \infty} \frac{x^2}{n^2 + x^2} = \lim_{n \to \infty} \frac{x^2}{1 + \frac{x^2}{n^2}} = \frac{0}{1 + 0} = 0.
\]

However, the convergence is not uniform. Negating the definition, \( g_n \) does not converge uniformly to \( g \) means there is an \( \epsilon_0 > 0 \) such that for every \( R > 0 \) there is an \( n \in \mathbb{N} \) such that \( n > R \) and there is an \( x \in \mathbb{R} \) such that \( |g_n(x) - g(x)| \geq \epsilon_0 \). Take \( \epsilon_0 = \frac{1}{2} \). Choose \( R \in \mathbb{R} \). Take \( n \in \mathbb{N} \) to satisfy \( n > R \) (by the Archimedean property) and let \( x = n \). Then
\[
|g_n(x) - g(x)| = |g_n(n) - 0| = \left| \frac{n^2}{n^2 + n^2} \right| = \frac{1}{2} \geq \epsilon_0. \]

[3.] Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.
(a.) Statement. Let \( \{y_n\} \) be a sequence such that \( y_n > 0 \) for all \( n \). If \( y_n \to y \) as \( n \to \infty \) then \( y > 0 \).

FALSE. Let \( y_n = \frac{1}{n} \). Then \( y_n \to 0 \) as \( n \to \infty \) but 0 is not positive.

(b.) Statement. Let \( \{z_n\} \) be a sequence that has a convergent subsequence. Then \( \{z_n\} \) is bounded.

FALSE. Let \( z_n = \begin{cases} 0, & \text{if } n \text{ is even;} \\ n, & \text{if } n \text{ is odd.} \end{cases} \) Then the even subsequence \( z_{2n} = 0 \to 0 \) but \( z_n \) is unbounded since \( z_{2n+1} = 2n + 1 \to \infty \) as \( n \to \infty \).

(c.) Statement. If \( f : (0,1) \to \mathbb{R} \) is uniformly continuous then \( \lim_{n \to \infty} f \left( \frac{1}{n} \right) \) exists.

TRUE. A uniformly continuous function on a bounded open interval has a continuous extension on the closure, \( F : [0,1] \to \mathbb{R} \) such that \( F = f \) on \( (0,1) \). But the sequence \( \{\frac{1}{n}\} \subset [0,1] \) tends to zero \( \frac{1}{n} \to 0 \) as \( n \to \infty \), and since \( F \) is continuous at zero, \( F(\frac{1}{n}) \to F(0) \) as \( n \to \infty \).
4. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Suppose that for some \( x_0 \in \mathbb{R} \) we have \( f(x_0) > 0 \). Show that there are \( a, b, c \in \mathbb{R} \) such that \( a < b \) and \( 0 < c \) such that \( f(x) \geq c \) whenever \( a < x < b \).

Proof. Since \( f \) is continuous at \( x_0 \), for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - f(x_0)| < \epsilon \) whenever \( |x - x_0| < \delta \). Apply this to the special case \( \epsilon = \frac{1}{2} f(x_0) > 0 \) and let \( \delta_0 > 0 \) be the corresponding \( \delta \). Then for \( x \in (x_0 - \delta_0, x_0 + \delta_0) \) we have \( |f(x) - f(x_0)| < \frac{1}{2} f(x_0) \). This implies for such \( x \),

\[
f(x) = f(x_0) + f(x) - f(x_0) \geq f(x_0) - |f(x) - f(x_0)| > f(x_0) - \frac{1}{2} f(x_0) = \frac{1}{2} f(x_0).
\]

Thus we have shown for \( a = x_0 - \delta_0 \), \( b = x_0 + \delta_0 \) and \( c = \frac{1}{2} f(x_0) \) that \( x \in (a, b) \) implies \( f(x) > c \).

5. Let \( f : \mathbb{R} \to \mathbb{R} \) and \( a, L \in \mathbb{R} \). State the definition: \( \lim_{x \to a} f(x) = L \). Using just the definition and not the limit theorems, show that \( \lim_{x \to 1} (x + 3)^2 = 16 \).

Definition. \( \lim_{x \to a} f(x) = L \) means for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( |f(x) - L| < \epsilon \) whenever \( x \in \mathbb{R} \) and \( 0 < |x - a| < \delta \).

Proof. Choose \( \epsilon > 0 \). Let \( \delta = \min\{1, \frac{\epsilon}{9}\} \). For any \( x \in \mathbb{R} \) that satisfies \( 0 < |x - 1| < \delta \) we have

\[
|x + 7| = |x - 1 + 8| \leq |x - 1| + 8 < \delta + 8 \leq 1 + 8 = 9 \text{ because } \delta \leq 1.
\]

Hence, \( \delta \leq \frac{\epsilon}{9} \) implies

\[
|f(x) - 16| = |(x + 3)^2 - (1 + 3)^2| = |(x + 3 + 1 + 3)(x + 3 - 1 - 3)| = |x + 7||x - 1| < 9\delta \leq \epsilon.
\]

\( \square \)