More Problems.

1. Suppose \( f : [-5, \infty) \to \mathbb{R} \) is defined by \( f(x) = \sqrt{5+x} \). Using just the definition of differentiability, show that \( f \) is differentiable at \( a = 5 \) and find \( f'(4) \).

   We show that the limit of the difference quotient exists.
   \[
   f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{5+x} - \sqrt{5+4}}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{5+x} - 3)(\sqrt{5+x} + 3)}{(x - 4)(\sqrt{5+x} + 3)} = \lim_{x \to 4} \frac{5 + x - 9}{x - 4} = \frac{1}{6}.
   \]

2. Suppose \( f : (a,b) \to \mathbb{R} \) is uniformly continuous. Show that the limit exists: \( \lim_{x \to b^-} f(x) \).

   Proof. i.e., a uniformly continuous function on \((a,b)\) has a continuous extension to \((a,b]\). This was a theorem in the text, but the problem asks us to prove it. Uniformly continuous means for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that \( |f(x) - f(y)| < \varepsilon \) whenever \( x, y \in (a,b) \) satisfy \( |x - y| < \delta \). Let \( \{x_n\} \) be any sequence in \((a,b)\) such that \( x_n \to a \) as \( n \to \infty \). We first show that \( \{f(x_n)\} \) is a Cauchy Sequence. To see it, choose \( \varepsilon > 0 \). By uniform continuity, there is a \( \delta > 0 \) so that \( |f(x_m) - f(x_k)| < \varepsilon \) whenever \( |x_m - x_k| < \delta \). But \( \{x_n\} \) is convergent, hence Cauchy. Thus there is an \( N \in \mathbb{R} \) so that \( |x_m - x_k| < \delta \) whenever any \( m, k \in \mathbb{N} \) satisfy \( m > N \) and \( k > N \). Hence \( |f(x_m) - f(x_k)| < \varepsilon \) whenever any \( m, k \in \mathbb{N} \) satisfy \( m > N \) and \( k > N \). But this says \( \{f(x_n)\} \) is a Cauchy Sequence.

   Since \( \{f(x_n)\} \) is Cauchy, it is convergent, so let \( L \in \mathbb{R} \) be the limit: \( f(x_n) \to L \) as \( n \to \infty \). We have found a subsequence converging to \( L \). The rest of the argument is to show that continuous limit \( f(x) \to L \) as \( x \to b^- \). To this end, we show that the definition of limit is satisfied: that for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that \( |f(x) - L| < \varepsilon \) for all \( x \in (a,b) \) such that \( b - \delta < x < b \). Choose \( \varepsilon > 0 \). By uniform convergence, there is a \( \delta > 0 \) so that \( |f(x) - f(y)| < \frac{\varepsilon}{2} \) whenever \( x, y \in (a,b) \) satisfy \( |x - y| < \delta \). This is the \( \delta \) needed for the limit. Now choose any \( x \in (a,b) \) such that \( b - \delta < x < b \). Since \( f(x_n) \to L \) as \( n \to \infty \), there is an \( N \in \mathbb{R} \) so that \( |f(x_n) - L| < \frac{\varepsilon}{2} \) whenever \( n > N \). Finally, since \( x_k \to b \) as \( k \to \infty \), there is a \( k \in \mathbb{N} \) so large that \( k > N \) and \( b - \delta < x_k < b \). By the usual sneaky adding and subtracting trick and the triangle inequality,
   \[
   |f(x) - L| = |f(x) - f(x_k) + f(x_k) - f(x)| \leq |f(x) - f(x_k)| + |f(x_k) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
   \]
   by uniform continuity since both \( x, x_k \in (b - \delta, b) \) so \( |x - x_k| < \delta \) and by the Cauchy of \( f(x_n) \). Thus \( f(x) \to L \) as \( x \to b^- \).

3. Give an example of a function \( f : \mathbb{R} \to \mathbb{R} \) that is differentiable at \( a = 0 \) but not for any other \( a \neq 0 \). Prove that your function has this property.

   Proof. We modify the Dirichlet function that is not continuous at any point. Let
   \[
   f(x) = \begin{cases} 
   x^2, & \text{if } x \in \mathbb{Q} \text{ is rational}, \\
   0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ is irrational}.
   \end{cases}
   \]
If $a \neq 0$ then $f$ is not continuous at $a$, hence not differentiable at $a$. Indeed by the density of rationals and irrationals, there are sequences $y_n \in \mathbb{Q}$ and $z_n \in \mathbb{R}\setminus\mathbb{Q}$ such that $y_n \to a$ and $z_n \to a$ as $n \to \infty$. Thus $f(y_n) = a^{2n} \to a^2$ as $n \to \infty$ but $f(z_n) = 0 \to 0$ as $n \to \infty$. Since two subsequences converging to $a$ result in inconsistent limits ($a^2 \neq 0$), the function $f$ is not continuous at $a$.

The differentiability at $a = 0$ follows because $f$ is squeezed between a “rock and a hard place.” For all $x \in \mathbb{R}$, $|f(x)| \leq x^2$. It follows that the difference quotient converges to zero. Indeed, choose $\epsilon > 0$ and let $\delta = \epsilon$. Then for any $x \in \mathbb{R}$, if $0 < |x - 0| < \delta$ then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{f(x)}{|x|} \right| \leq \frac{|x|^2}{|x|} = |x| < \delta = \epsilon.$$ 

Thus $f$ is differentiable at $a = 0$ since the limit exists: $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$.

4. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which is differentiable on $(0, \infty)$. Suppose that $f(0) = 0$ and $|f'(x)| < M$ for all $x \in (0, \infty)$. Show that for all $x \geq 0$,

$$|f(x)| \leq M|x|.$$ 

**Proof.** If $x = 0$ then $|f(0)| = |0| \leq M|0|$. Thus suppose $x > 0$. Because $f$ is continuous on $[0, x]$ and differentiable on $(0, x)$, my the mean value theorem, there is $c \in (0, x)$ so that

$$|f(x)| = |f(x) - f(0)| = |f'(c)(x - 0)| = |f'(c)||x| \leq M|x|$$

because $|f'(c)| \leq M$ holds for any $c > 0$.

5. Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable on $\mathbb{R}$ but $f'(x)$ is not continuous on $\mathbb{R}$. Prove that your function has this property.

**Proof.** Let

$$f(x) = \begin{cases} 
    x^2 \sin \left( \frac{1}{x} \right), & \text{if } x \neq 0, \\
    0, & \text{if } x = 0.
\end{cases}$$

For all $x \in \mathbb{R}$, this function is squeezed $|f(x)| \leq x^2$. As in problem (3), $f$ is differentiable at zero and $f'(0) = 0$. For $x \neq 0$, the function is the product and composition of differentiable functions, whose derivative is gotten by the product and chain rules

$$f'(x) = 2x \sin \left( \frac{1}{x} \right) - \cos \left( \frac{1}{x} \right).$$

For the sequence $x_n = \frac{1}{2n\pi}$, which tends to zero, we have $f'(x_n) = -1$ so that $f'(x_n) \to -1$ as $n \to \infty$. As this is not $f'(0) = 0$, $f'$ is not continuous at 0.

6. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable on $(a, b)$. Suppose $x, y \in (a, b)$ and that $m$ is any number between $f'(x)$ and $f'(y)$. Then there is a $z$ between $x$ and $y$ such that $f'(z) = m$. In other words, the mean value property holds for the derivative, even though the derivative may not be a continuous function.

**Proof.** Choose $x, y \in (a, b)$. For convenience, let us assume that $x < y$ and $f'(x) < m < f'(y)$. Other cases are similar. Then the function $g(x) = f(x) - mx$ is continuous on $[x, y]$ and differentiable on $(a, b)$. Also $g'(x) = f'(x) - m < 0$ and $g'(y) = f'(y) - m > 0$.

As in the homework problem, it follows that for $z \in (x, y)$ close enough to $x$, $g(z) < g(x)$ and for $z$ close enough to $y$, $g(z) < g(y)$. To see this, let’s do the $y$ case. Since $g$ is differentiable at $y$,

$$g'(y) = \lim_{z \to y} \frac{g(z) - g(y)}{z - y}.$$
so for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) so that for any \( z \in (x, y) \) so that \( y - \delta < z < y \) we have

\[
\left| \frac{g(z) - g(y)}{z - y} - g'(y) \right| < \varepsilon.
\]

Applying this to \( \epsilon = g'(y) > 0 \), if \( z \in [x, y] \) satisfies \( y - \delta < z < y \) then

\[
g(z) = g(y) + \left( \frac{g(z) - g(y)}{z - y} - g'(y) \right) (z - y) + g'(y)(z - y)
\leq g(y) + \left| \frac{g(z) - g(y)}{z - y} - g'(y) \right| |z - y| + g'(y)(z - y)
< g(y) + g'(y)|z - y| + g'(y)(z - y) = g(y).
\]

Thus it follows that there are points in the interval \( z_i \in [x, y] \) such that \( g(z_1) < g(x) \) and \( g(z_2) < g(y) \). But since \( g \) is continuous on \([x, y]\), by the minimum theorem, there is \( c \in [x, y] \) so that

\[
g(c) = \inf_{z \in [x, y]} g(z).
\]

But \( c \) cannot be the endpoint because \( g(c) \leq \min\{g(z_1), g(z_2)\} < \min\{g(x), g(y)\} \), thus \( c \in (x, y) \), where \( g \) is differentiable. It follows from the theorem about the vanishing of the derivative at a minimum point and the definition of \( g \),

\[
f'(c) - m = g'(c) = 0
\]

so that at the intermediate point \( f'(c) = m \), as desired.

7. Which is bigger \( e^\pi \) or \( \pi^e \)?

**Proof.** Consider the function \( f(x) = e^{-x}x^e \). It is continuous on \([0, \infty)\) and differentiable on \((0, \infty)\) since we use logs and exponentials to define \( f(x) = \exp(g(x)) \) where \( g(x) = -x + e \ln x \). Since \( g'(x) = -1 + \frac{x}{e} \), it follows that \( g'(x) > 0 \) for \( x \in (0, e) \) and \( g'(x) < 0 \) for \( x \in (e, \infty) \). Hence \( g(e) = 0 > g(\pi) \) by the corollary to the Mean Value Theorem relating decreasing to derivatives. Because \( \exp \) is a strictly increasing function, it is increasing and decreasing on the same intervals as \( g \). It follows from \( e < \pi \) that

\[
1 = f(e) = \exp(g(e)) = \exp(0) > \exp(g(\pi)) = f(\pi) = \frac{\pi^e}{e^\pi}
\]

so \( \pi^e < e^\pi \).

8. Suppose that \( f : (a, \infty) \to \mathbb{R} \) is differentiable and that \( f'(x) \to \infty \) as \( x \to \infty \). Then show that \( f \) is not uniformly continuous on \((0, \infty)\).

**Proof.** By negating the definition, we are to show \( f \) is not uniformly continuous on \((a, \infty)\) which means there exists an \( \varepsilon > 0 \) such that for every \( \delta > 0 \) there are \( x, y \in (a, \infty) \) such that \( |x - y| < \delta \) and \( |f(x) - f(y)| \geq \varepsilon \). We show this is true for \( \epsilon = 1 \). Choose \( \delta > 0 \). Since \( f'(x) \to \infty \), there is \( R \in \mathbb{R} \) so that \( f'(c) > \frac{\delta}{2} \) whenever \( c \in (a, \infty) \) satisfies \( c > R \). Now pick an \( x \in (a, \infty) \) such that \( x > R \). Let \( y = x + \frac{\delta}{2} \). We have \( y \in (a, \infty) \) and it satisfies \( |y - x| = \frac{\delta}{2} < \delta \). Because \( f \) is continuous on \([x, y]\) and differentiable on \((x, y)\), by the Mean Value Theorem, there is a \( c \in (x, y) \) such that

\[
|f(y) - f(x)| = |f'(c)(y - x)| = f'(c)(y - x) > \frac{2 \cdot \delta}{2} = 1 \geq \varepsilon
\]

because \( c > x > R \).
9. Let \( f : [a, b] \to \mathbb{R} \) be an integrable function. Then \( f^2 \) is integrable on \([a, b]\) and

\[
\left( \int_a^b f(x) \, dx \right)^2 \leq (b-a) \int_a^b f^2(x) \, dx.
\]

Proof. First, for all \( x \in [a, b] \), \( \inf_{x \in [a, b]} f \leq f(x) \leq \sup_{x \in [a, b]} f \) so \( 0 \leq f^2(x) \leq M \) where \( M = \max\{|\inf_{x \in [a, b]} f^2|, |\sup_{x \in [a, b]} f^2|\} \). Thus \( f \) is bounded.

Next we show \( f^2 \) is integrable using the theorem that says \( g \) is integrable on \([a, b]\) if and only if for every \( \varepsilon > 0 \) there is a partition \( \mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\} \) of \([a, b]\) so that \( U(g, \mathcal{P}) - L(g, \mathcal{P}) < \varepsilon \). Here the upper and lower sums are

\[
U(g, \mathcal{P}) = \sum_{k=1}^{n} M_k(g)(x_k - x_{k-1}), \quad L(g, \mathcal{P}) = \sum_{k=1}^{n} m_k(g)(x_k - x_{k-1})
\]

where

\[
M_k(g) = \sup_{x \in [x_{k-1}, x_k]} g(x), \quad m_k(g) = \inf_{x \in [x_{k-1}, x_k]} g(x).
\]

For any partition, consider three cases for a subinterval: \( M_k \leq 0, m_k \leq 0 \leq M_k \) and \( 0 \leq m_k \). In the first case for \( x \in [x_{k-1}, x_k] \),

\[
m_k(f) \leq f(x) \leq M_k(f) \leq 0
\]

so that

\[
0 \leq M_k(f)^2 - m_k(f)^2 \leq f^2(x) \leq m_k(f)^2
\]

which implies

\[
M_k(f^2) - m_k(f^2) \leq m_k(f)^2 - M_k(f)^2 = |m_k(f) + M_k(f)||m_k(f) - M_k(f)| \leq 2M(M_k(f) - m_k(f)).
\]

In the second case for \( x \in [x_{k-1}, x_k] \), \( m_k(f) \leq 0 \leq M_k(f) \). Thus,

\[
m_k(f) \leq f(x) \leq M_k(f)
\]

so that

\[
0 \leq f^2(x) \leq \max\{m_k(f)^2, M_k(f)^2\} \leq (M_k(f) - m_k(f))^2 \leq (|M_k(f)| + |m_k(f)|)(M_k(f) - m_k(f))
\]

which implies

\[
M_k(f^2) - m_k(f^2) \leq M_k(f^2) \leq 2M(M_k(f) - m_k(f)).
\]

In the third case, for \( x \in [x_{k-1}, x_k] \),

\[
0 \leq m_k(f) \leq f(x) \leq M_k(f)
\]

so that

\[
0 \leq m_k(f)^2 \leq f^2(x) \leq M_k(f)^2
\]

which implies

\[
M_k(f^2) - m_k(f^2) \leq M_k(f^2) - m_k(f)^2 = |M_k(f) + m_k(f)||M_k(f) - m_k(f)| \leq 2M(M_k(f) - m_k(f)).
\]

Hence, for any subinterval \([x_{k-1}, x_k]\) in every case we have

\[
M_k(f^2) - m_k(f^2) \leq 2M(M_k(f) - m_k(f)).
\]
To prove $f^2$ is integrable, choose $\varepsilon > 0$. Since $f$ is integrable, there is a partition $P$ such that
\[ U(f, P) - L(f, P) < \frac{\varepsilon}{2M + 1}. \]
Then the same partition applied to $f^2$ yields
\[
U(f^2, P) - L(f^2, P) = \sum_{k=1}^{n} (M_k(f^2) - m_k(f^2)) (x_k - x_{k-1}) \\
\leq \sum_{k=1}^{n} 2M(M_k(f) - m_k(f)) (x_k - x_{k-1}) \\
= 2M(U(f, P) - L(f, P)) \\
< \frac{2M\varepsilon}{2M + 1} < \varepsilon
\]
Thus $f^2$ is integrable.

Inequality (1) is known as the Schwartz Inequality. Its proof is a little trick. The inequality is trivial if $a = b$, so we assume $a < b$. For each $t \in \mathbb{R}$ the function $(f(x) + t)^2$ is integrable since it is the square of an integrable function $f(x) + t$. It is nonnegative, thus for all $t \in \mathbb{R}$,
\[
0 \leq \int_a^b (f(x) + t)^2 \, dx = \int_a^b f(x)^2 \, dx + 2t \int_a^b f(x) \, dx + t^2 \int_a^b dx = \alpha + 2\beta t + \gamma t^2.
\]
The quadratic function is minimized when $t = -\frac{\beta}{\gamma}$. Substituting this $t$,
\[
0 \leq \alpha - \frac{2\beta^2}{\gamma} + \frac{\gamma\beta^2}{\gamma^2} = \alpha - \frac{\beta^2}{\gamma} = \int_a^b f(x)^2 \, dx - \frac{1}{b-a} \left( \int_a^b f(x) \, dx \right)^2
\]
which is the Schwartz Inequality (1).

10. Let $f_n, f : [a, b] \to \mathbb{R}$ be functions defined on a closed, bounded interval. Assume that $f_n$ are bounded and integrable, and that $f_n \to f$ uniformly as $n \to \infty$. Then $f$ is integrable and we can interchange limit and integral
\[
\lim_{n \to \infty} \left( \int_a^b f_n(x) \, dx \right) = \int_a^b f(x) \, dx. \tag{2}
\]

Proof. First, we show $f$ is bounded. Since $f_n \to f$ converges uniformly, for every $\varepsilon > 0$, there is an $N(\varepsilon) \in \mathbb{N}$ so that if $n \in \mathbb{N}$ satisfies $n > N(\varepsilon)$ and $x \in [a, b]$, then $|f_n(x) - f(x)| < \varepsilon$. Taking $\varepsilon = 1$, and fix an $n \in \mathbb{N}$ large enough so $n > N(1)$, then any $x \in [a, b]$ satisfies
\[
|f(x)| = |f_n(x) + f(x) - f_n(x)| \leq |f_n(x)| + |f(x) - f_n(x)| \leq \sup_{x \in [a, b]} |f_n(x)| + 1
\]
which is finite because $f_n$ is bounded. Thus $f$ is bounded.

Next we show $f$ is integrable. For this purpose, we use the theorem that says $f$ is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ so that $U(f, P) - L(f, P) < \varepsilon$. Here the upper and lower sums are
\[
U(f, P) = \sum_{k=1}^{n} M_k(f) (x_k - x_{k-1}), \quad L(f, P) = \sum_{k=1}^{n} m_k(f) (x_k - x_{k-1})
\]
where
\[ M_k(f) = \sup_{x \in [x_{k-1}, x_k]} f(x), \quad m_k(f) = \inf_{x \in [x_{k-1}, x_k]} f(x). \]

Now choose \( \varepsilon > 0 \). We approximate \( f \) by an \( f_\ell \), then choose a partition that is good for \( f_\ell \) and then show it is good for \( f \). Since the convergence is uniform, there is \( N \in \mathbb{R} \) so that whenever \( \ell \in \mathbb{N} \) satisfies \( \ell > N \) and every \( x \in [a, b] \) we have
\[ |f_\ell(x) - f(x)| < \frac{\varepsilon}{6(b-a) + 6}. \] 

(3)

We pick one such \( \ell \) to show integrable. Thus, for every \( x \in [a, b] \),
\[ f_\ell(x) - \frac{\varepsilon}{6(b-a) + 6} < f(x) < f_\ell(x) + \frac{\varepsilon}{6(b-a) + 6}. \]

Now \( f_\ell \) is integrable, so by the theorem, there is a partition of \( \mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\} \) of \([a, b]\) so that \( U(f_\ell, \mathcal{P}) - L(f_\ell, \mathcal{P}) < \frac{\varepsilon}{5} \). Hence taking inf and sup over \([x_{k-1}, x_k]\),
\[ m_k(f_\ell) - \frac{\varepsilon}{6(b-a) + 6} \leq \inf_{x \in [x_{k-1}, x_k]} f_\ell - \frac{\varepsilon}{6(b-a) + 6} \leq \inf_{x \in [x_{k-1}, x_k]} f = m_k(f), \]
\[ M_k(f) = \sup_{x \in [x_{k-1}, x_k]} f \leq \sup_{x \in [x_{k-1}, x_k]} f_\ell + \frac{\varepsilon}{6(b-a) + 6} \leq M_k(f_\ell) + \frac{\varepsilon}{6(b-a) + 6}. \]

It follows that
\[ M_k(f) - m_k(f) \leq M_k(f_\ell) - m_k(f_\ell) + \frac{\varepsilon}{3(b-a) + 3}. \]

Summing over the subintervals,
\[ U(f, \mathcal{P}) - L(f, \mathcal{P}) = \sum_{k=1}^{n} (M_k(f) - m_k(f)) (x_k - x_{k-1}) \]
\[ \leq \sum_{k=1}^{n} \left( M_k(f_\ell) - m_k(f_\ell) + \frac{\varepsilon}{3(b-a) + 3} \right) (x_k - x_{k-1}) \]
\[ = U(f_\ell, \mathcal{P}) - L(f_\ell, \mathcal{P}) + \frac{\varepsilon}{3(b-a) + 3} \sum_{k=1}^{n} (x_k - x_{k-1}) \]
\[ < \frac{\varepsilon}{3} + \frac{\varepsilon(b-a)}{3(b-a) + 3} < \varepsilon. \]

Hence \( f \) is integrable.

To show that the limit of the integrals is the integral of the limit, choose \( \epsilon > 0 \) and let \( N \in \mathbb{R} \) as above. Applying (3), for every \( n > N \) we get
\[ \left| \int_{a}^{b} f(x) \, dx - \int_{a}^{b} f_n(x) \, dx \right| \leq \int_{a}^{b} |f(x) - f_n(x)| \, dx \leq \int_{a}^{b} \frac{\varepsilon \, dx}{6(b-1) + 6} = \frac{\varepsilon(b-a)}{6(b-1) + 6} < \varepsilon. \]

Thus we have shown (2).

11. Define \( \log x = \int_{1}^{x} \frac{dt}{t} \) for \( x > 0 \) as usual. If \( x = e^y \) is the inverse function of \( y = \log x \), show that \( e^y \) is differentiable and \( \frac{d}{dy} e^y = e^y \).

The differentiability of \( F(x) = \log x \) follows from the Fundamental Theorem of Calculus, which says that if \( f \) is integrable on \([a, b]\) for any \( 0 < a < b < \infty \) then \( F(z) = \int_{a}^{z} f(t) \, dt \).
is uniformly continuous on \([a, b]\) and if \(f\) is continuous at \(z \in (a, b)\), then \(F\) is differentiable at \(z\) and \(F'(z) = f(z)\). In our case \(f(t) = \frac{1}{t}\) so it is continuous, hence integrable on \([a, b]\) and since \(f\) is continuous at \(z\), \(F'(z) = \frac{1}{z}\).

Since \(F'(z) > 0\), it is strictly increasing, and since \(F(z)\) is continuous, the inverse function theorem for continuous functions says that \(F^{-1} = \exp : [\log a, \log b] \to \mathbb{R}\) is continuous and strictly increasing. We have defined \(F\) on \((0, \infty)\) (by taking \(a\) small and \(b\) large enough). So choose \(w \in F((0, \infty))\) and let \(F(z) = w\) be the corresponding point inverse to \(w\). By the theorem on derivatives of inverse functions, which says, if \(F\) is monotone on \((0, \infty)\) and differentiable at \(z \in (0, \infty)\) and \(F'(z) = \frac{1}{z} \neq 0\), then the inverse function is differentiable at \(w = F(z)\) and

\[
\frac{d}{dy} e^y \bigg|_{y=w} = \frac{d}{dy} F^{-1}(y) \bigg|_{y=w} = \frac{1}{F'(z)} = \frac{1}{1/z} = z = F^{-1}(w) = e^w.
\]

12. **Does the improper integral** \(\int_{-\infty}^{\infty} \frac{dt}{(t^2 + t^4)^{\frac{3}{2}}}\) **converge? Why?**

There are four limits: at \(-\infty, 0-, 0+, \) and \(\infty\). Split the integral into four parts

\[
I_1 + I_2 + I_3 + I_4 = \int_{-\infty}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{\infty}
\]

Use the comparison theorem for improper integrals. If \(f, g\) are integrable on all subintervals and if \(|f(t)| \leq g(t)\) for all \(t\) and if the improper integral \(\int f(t)\ dt\) converges then the improper integral \(\int g(t)\ dt\) converges. For the interval \(I_2\) and \(I_3\), we have for \(0 < |t| \leq 1\),

\[
|f(t)| = \frac{1}{(t^2 + t^4)^{\frac{3}{2}}} \leq \frac{1}{t^3} = g(t)
\]

and the improper integral converges

\[
\int_{0}^{1} g(t)\ dt = \int_{0}^{1} \frac{dt}{t^\frac{3}{2}} = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{dt}{t^\frac{3}{2}} = \lim_{\varepsilon \to 0^+} \left[3t^{\frac{1}{2}}\right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0^+} \left[3 - 3\varepsilon^{\frac{3}{2}}\right] = 3.
\]

Thus the improper integral \(I_3\) exists. Because \(g(t)\) is an even function

\[
\int_{-1}^{0} g(t)\ dt = \int_{-1}^{0} \frac{dt}{t^\frac{3}{2}} = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{1} \frac{dt}{t^\frac{3}{2}} = \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{dt}{t^\frac{3}{2}} = 3
\]

from before. Thus the improper integral \(I_2\) exists also.

For the interval \(I_1\) and \(I_4\), we have for \(1 \leq |t|\),

\[
|f(t)| = \frac{1}{(t^2 + t^4)^{\frac{3}{2}}} \leq \frac{1}{t^3} = g(t)
\]

and the improper integral converges

\[
\int_{1}^{\infty} g(t)\ dt = \int_{1}^{\infty} \frac{dt}{t^\frac{3}{2}} = \lim_{R \to \infty} \int_{1}^{R} \frac{dt}{t^\frac{3}{2}} = \lim_{R \to \infty} \left[-3t^{-\frac{1}{2}}\right]_{1}^{R} = \lim_{R \to \infty} \left[3 - 3R^{-\frac{3}{2}}\right] = 3.
\]

Thus the improper integral \(I_4\) exists. Because \(g(t)\) is an even function

\[
\int_{-\infty}^{-1} g(t)\ dt = \int_{-\infty}^{-1} \frac{dt}{t^\frac{3}{2}} = \lim_{R \to \infty} \int_{-R}^{-1} \frac{dt}{t^\frac{3}{2}} = \lim_{R \to \infty} \int_{1}^{R} \frac{dt}{t^\frac{3}{2}} = 3
\]

from before. Thus the improper integral \(I_1\) exists also.
13. Show that if the limit \( \lim_{n \to \infty} \frac{|b_k|}{|a_k|} = L \) exists and if \( \sum a_k \) converges absolutely then \( \sum b_k \) converges absolutely.

The existence of the limit of nonnegative numbers so \( L \geq 0 \) shows that the series can be compared. There is an \( N \in \mathbb{R} \) so that \( a_k \neq 0 \) and

\[
\frac{|b_k|}{|a_k|} < L + 1
\]

whenever \( k > N \). Hence, for all \( k > N \),

\[
|b_k| \leq (L + 1)|a_k|.
\]

Hence, by the regular comparison test, \( \sum_{k=1}^{\infty} |b_k| \) is convergent because \( \sum_{k=1}^{\infty} (L + 1)|a_k| \) is convergent by assumption.

14. Determine whether \( \sum_{k=1}^{\infty} (-1)^k \frac{(k!)^2}{(2k)!} \) is absolutely convergent, conditionally convergent or divergent.

To check absolute convergence, use the ratio test.

\[
\rho = \lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k \to \infty} \frac{(k+1)(k+1)!}{(2k+2)(2k+1)} \cdot \frac{2k+1}{k+1} = \lim_{k \to \infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4}.
\]

Since \( \rho < 1 \), the series is absolutely convergent.

15. Let \( a^+ = a \) if \( a \geq 0 \) and \( a^+ = 0 \) if \( a < 0 \). Similarly, let \( a^- = \min\left\{0, a\right\} \). Show that if \( A = \sum_{k=1}^{\infty} a_k \) is conditionally convergent, then the series

\[
P = \sum_{k=1}^{\infty} a^+_k, \quad M = \sum_{k=1}^{\infty} a^-_k
\]

are both divergent.

Argue by contradiction. We assume that \( A \) is conditionally convergent and both \( P \) and \( M \) are not divergent. Thus we may assume that one of the sums, say \( P \), is convergent. Using the fact that \( a_k = a^+_k + a^-_k \), we have the series of differences from two convergent series is convergent and converges to the difference, so

\[
M = \sum_{k=1}^{\infty} a^-_k = \sum_{k=1}^{\infty} (a_k - a^+_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} a^+_k = A - P
\]

converges also. Similarly if \( M \) converges then so does \( P \). Thus both \( P \) and \( M \) converge. It follows that \( A \) is absolutely convergent. This is because \( |a_k| = a^+_k - a^-_k \). Again, the convergence of the sum of differences follows from the convergence of the individual series

\[
\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} (a^+_k - a^-_k) = \sum_{k=1}^{\infty} a^+_k - \sum_{k=1}^{\infty} a^-_k = P - M.
\]

Thus we have shown that \( A \) is absolutely convergent, hence not conditionally convergent.
16. Show that if \( A = \sum_{k=1}^{\infty} a_k \) is convergent and \( \{b_k\} \) is bounded and monotone, then \( A = \sum_{k=1}^{\infty} a_k b_k \) is convergent. (This is known as Abel’s Test.)

This one relies on a trick called Abel’s Summation by Parts. If \( S_k = \sum_{j=1}^{k} a_j \) is the partial sum, then

\[
\sum_{k=1}^{n} a_k b_k = S_n b_{n+1} + \sum_{k=1}^{n} S_k (b_k - b_{k+1}).
\]

(4)

To see this, observe that for \( k \in \mathbb{N} \),

\[
a_k b_k = (S_k - S_{k-1}) b_k = S_k (b_k - b_{k+1}) + (S_k b_{k+1} - S_{k-1} b_k)
\]

where we understand \( S_0 = 0 \). Now summing gives (4), noting that the second parenthesized term telescopes.

Since \( \{b_n\} \) is bounded and monotone, it is convergent: \( b_n \to B \) as \( n \to \infty \). Since \( A \) is convergent, \( S_n \to A \) as \( n \to \infty \). Thus the first term in the partial sum (4) converges to a limit \( S_n b_{n+1} \to AB \) as \( n \to \infty \).

The sum \( B = \sum_{k=1}^{\infty} (b_k - b_{k-1}) \) is convergent because \( b_n = \sum_{k=1}^{n} (b_k - b_{k-1}) \to B \) as \( n \to \infty \) where we have taken \( b_0 = 0 \). Since \( \{b_n\} \) is monotone, the summands have a fixed sign and the convergence is absolute. Finally, since \( S_n \to A \) as \( n \to \infty \), it is bounded. This implies that the last sum in (4) converges.

To show \( T_n = \sum_{k=1}^{n} S_k (b_k - b_{k+1}) \) tends to a limit as \( n \to \infty \), suppose the bound is \( |S_k| \leq M \) for all \( k \). Now we check the Cauchy Criterion. Choose \( \varepsilon > 0 \). By the convergence of \( \{b_n\} \), there is an \( N \in \mathbb{R} \) so that \( m, \ell > N \) implies \( |b_{m+1} - b_{\ell+1}| < \frac{\varepsilon}{M+1} \). So for any \( m, \ell > N \) so that \( \ell > m \),

\[
|T_m - T_\ell| = \left| \sum_{k=m+1}^{\ell} S_k (b_k - b_{k+1}) \right| \leq \sum_{k=m+1}^{\ell} |S_k| |b_k - b_{k+1}|
\]

\[
\leq \sum_{k=m+1}^{\ell} M |b_k - b_{k+1}| = \left| \sum_{k=m+1}^{\ell} M (b_k - b_{k+1}) \right| = M |b_{m+1} - b_{\ell+1}| < \varepsilon.
\]

Thus we have shown \( \{T_n\} \) is Cauchy, hence convergent.