Final Given Dec. 15, 2000. (That course covered Chapter 7 instead of Chapter 5.)

1. Using only the definition of differentiability and limit theorems, show that \( f(x) = \frac{x}{1+x^2} \) is differentiable at \( x = 2 \) and that \( f'(2) = \frac{1}{4} \).

2. Let \( E = \left\{ -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots \right\} = \left\{ \frac{(-1)^n n}{n+1} : n = 1, 2, 3, \ldots \right\} \). Find \( \inf E \). Prove your answer.

3. Define a sequence by \( x_1 = 10 \) and \( x_{n+1} = 2 + \frac{1}{2}x_n \) for \( n \geq 1 \).
   i. Show that \( x_n \) is decreasing.
   ii. Show that \( x_n \) is bounded below.
   iii. Show that \( x_n \to 3 \) as \( n \to \infty \).

4. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a sequence of real numbers.
   i. State the definition: \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy Sequence.
   ii. Show that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is a Cauchy Sequence, where
   \[
   y_1 = \frac{1}{1}, \quad y_2 = \frac{1}{1} - \frac{1}{1+\frac{1}{2}}, \quad y_3 = \frac{1}{1} - \frac{1}{1+\frac{1}{2}+\frac{1}{2}}, \quad y_4 = \frac{1}{1} - \frac{1}{1+\frac{1}{2}+\frac{1}{3}}, \quad y_5 = \frac{1}{1} - \frac{1}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}}, \ldots
   \]
   In general, \( y_n = \sum_{k=1}^{n} \frac{(-1)^{n+1}}{k!} \),\[
   \text{[Hint: You may first wish to prove that } \frac{1}{n!} \leq \frac{1}{2^{n-1}}.\]

5. Let \( E \subset \mathbb{R} \) be a subset, \( a \in E \) be a point and \( f : E \to \mathbb{R} \) be a function.
   i. State the definitions: \( f \) is continuous at \( a \). \( f \) is continuous on \( E \).
   ii. Define \( f(x) = x + \frac{1}{2} \). Show directly from the definition that \( f \) is continuous on \( (0, \infty) \).

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \text{ (} x \text{ is a rational number,)} \\ 0, & \text{if } x \notin \mathbb{Q} \text{ (} x \text{ is not a rational number.)} \end{cases} \)
   i. Show that \( f \) is differentiable at \( x = 0 \) and find \( f'(0) \).
   ii. If \( a \neq 0 \), is \( f \) differentiable at \( a \)? Why?

7. Assume that the function \( f : \mathbb{R} \to \mathbb{R} \) is differentiable on \( \mathbb{R} \) and satisfies \( f(x) > 0 \) for all \( x \in \mathbb{R} \). Using only the definition of differentiability and limit theorems (and not the quotient rule for derivatives,) show that the reciprocal \( g(x) = \frac{1}{f(x)} \) is differentiable for all \( a \in \mathbb{R} \) and that \( g'(a) = -\frac{f''(a)}{f(a)^2} \).

8. Define a function \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \frac{x^3 + x^2 - 2x}{x^2 + 1} \). Give an argument if true; give a counterexample if false:
   i. \( f : \mathbb{R} \to \mathbb{R} \) is onto.
   ii. \( f : \mathbb{R} \to \mathbb{R} \) is one-to-one.
   iii. There is a number \( x_0 \in \mathbb{R} \) so that \( f'(x_0) = 0 \).

9. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Suppose there is a constant \( 0 \leq M < \infty \) so that \( |f'(\xi)| \leq M \) for all \( \xi \in \mathbb{R} \).
   i. Prove that for all \( x, y \in \mathbb{R} \) there holds \( |f(x) - f(y)| \leq M|x - y| \).
   ii. Assuming (i.), prove that \( f \) is uniformly continuous on \( \mathbb{R} \).

10. Define the sequence of functions by \( f_n(x) = \frac{1}{n^2} + \frac{x}{n} \). Find \( \lim_{n \to \infty} f_n(x) \) and show that the convergence is uniform on \([-3, 3]\).

1. Let \( E = \left\{ \frac{p}{q} : p, q \in \mathbb{N} \right\} \). Find the infimum, \( \inf E \). Prove your answer.

2. Using only the definition of integrability, prove that \( f(x) \) is integrable on \([0,1]\), where

\[
f(x) = \begin{cases} 
0, & \text{if } x = \frac{1}{3}; \\
1, & \text{if } x = \frac{2}{3}; \\
2, & \text{otherwise}.
\end{cases}
\]

3. Let \( f : [0,1] \rightarrow \mathbb{R} \) be continuous on \([0,1]\) and suppose that \( f(x) = 0 \) for each rational number \( x \) in \([0,1]\). Prove that \( f(x) = 0 \) for all \( x \in [0,1] \).

4. Determine whether the statements are true or false. If the statement is true, give the reason. If the statement is false, provide a counterexample.

   i. **Statement.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be differentiable at \( a \). Then \( \lim_{h \to 0} \frac{f(a + h) - f(a - h)}{2h} = f'(a) \)

   ii. **Statement.** Let \( f : [0,1] \rightarrow \mathbb{R} \) be such that \( |f(x)| \) is Riemann integrable on \([0,1]\). Then \( f(x) \) is Riemann Integrable on \([0,1]\).

   iii. **Statement.** If \( f : [a,b] \to \mathbb{R} \) is differentiable on \([a,b]\). Then \( F(x) = \int_a^x f(t)\,dt \) is continuous on \([a,b]\).

5. Suppose that \( f \) and \( g \) are continuous functions on \([0,1]\) and differentiable on \((0,1)\). Suppose that \( f(0) = g(0) \) and that \( f'(x) \leq g'(x) \) for all \( x \in (0,1) \). Show that \( f(x) \leq g(x) \) for all \( x \in [0,1] \).

6. Let \( E \subseteq \mathbb{R} \) and \( f : E \rightarrow \mathbb{R} \).

   i. State the definition: \( f \) is uniformly continuous on \( E \).

   ii. Let \( f(x) \) be uniformly continuous on \( \mathbb{R} \). Prove that \( \lim_{t \to 0} \left\{ \sup_{x \in \mathbb{R}} |f(x) - f(x + t)| \right\} = 0 \).

7. Show that \( \{z_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence, where \( z_n = \int_n^{n+1} \frac{\sin t}{1 + t} \,dt \).

8. Let \( x_1 = 0 \) and \( x_{n+1} = \frac{1}{2} + \sin(x_n) \) for all \( n > 1 \). Prove that \( \{x_n\}_{n \in \mathbb{N}} \) converges.
Some solved problems from the last quarter of the semester (since the last midterm).

(1.) Show that \( f(x) = [3x] \) is integrable on \([0,1]\), where \([x]\) is the greatest integer function. What is \( \int_0^1 f(x) \, dx ? \)

To show \( f \) is integrable on \([0,1]\) we need to show that for every \( \varepsilon > 0 \) there is a partition \( P = \{0 = x_0 < x_1 < \cdots < x_n = 1\} \) of \([0,1]\) so that the upper sum minus the lower sum satisfies \( U(f,P) - L(f,P) < \varepsilon \).

Choose \( \varepsilon > 0 \) and let \( \eta \) be any number such that \( 0 < \eta < \min\{\varepsilon,1\} \). Because \( f(x) = 0 \) on the interval \([0,\frac{1}{3})\), \( f = 1 \) on \([\frac{1}{3},\frac{1}{2})\), \( f = 2 \) on \([\frac{2}{3},1)\) and \( f(1) = 3 \), we shall choose a partition that has narrow intervals near the jumps of the function. Let

\[
P = \left\{ x_0 = 0 < x_1 = \frac{1-\eta}{3} < x_2 = \frac{1}{3} < x_3 = \frac{2-\eta}{3} < x_4 = \frac{2}{3} < x_5 = \frac{3-\eta}{3} < x_6 = 1 \right\}
\]

The sups and infs on the intervals are computed as follows

\[
M_1 = \sup_{x \in [x_0, x_1]} f(x) = \sup_{x \in [0, \frac{1-\eta}{3}]} 0 = 0, \quad M_2 = \sup_{x \in [\frac{1-\eta}{3}, \frac{1}{3}]} f(x) = 1, \quad M_3 = 1, \quad M_4 = 2, \quad M_5 = 2, \quad M_6 = 3;
\]

\[
m_1 = \inf_{x \in [x_0, x_1]} f(x) = \inf_{x \in [0, \frac{1-\eta}{3}]} 0 = 0, \quad m_2 = \inf_{x \in [\frac{1-\eta}{3}, \frac{1}{3}]} f(x) = 0, \quad m_3 = 1, \quad m_4 = 1, \quad m_5 = 2, \quad m_6 = 2;
\]

Hence, since \( M_1 - m_1 = M_3 - m_3 = M_5 - m_5 = 0 \), \( M_2 - m_2 = M_4 - m_4 = M_6 - m_6 = 1 \) and \( x_{2j} - x_{2j-1} = \frac{\eta}{3} \),

\[
U - L = \sum_{i=1}^{6} (M_i - m_i)(x_i - x_{i-1}) = \frac{\eta}{3} [(M_2 - m_2) + (M_4 - m_4) + (M_6 - m_6)] = \frac{3\eta}{3} < \varepsilon.
\]

Thus \( f \) is integrable on \([0,1]\). We may compute the upper sum using \( x_{2j+1} - x_{2j} = \frac{1-\eta}{3} \)

\[
U = \sum_{i=1}^{6} M_i(x_i - x_{i-1}) = 0 + 1 \cdot (x_2 - x_1) + 1 \cdot (x_3 - x_2) + 2 \cdot (x_4 - x_3) + 2 \cdot (x_5 - x_4) + 3 \cdot (x_6 - x_5)
\]

\[
= \frac{\eta}{3} + \frac{1-\eta}{3} + 2 \cdot \frac{\eta}{3} + 2 \cdot \frac{1-\eta}{3} + 3 \cdot \frac{\eta}{3} = 1 + \eta
\]

Similarly \( L(f,P) = 1 \). We deduce the value of the integral from the fact that for integrable functions, for any partition the upper and lower sums bracket the integral. Thus for our partition above,

\[
1 = L(f,P) \leq \int_0^1 f(x) \, dx \leq U(f,P) < 1 + \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, \( \int_0^1 f(x) \, dx = 1 \).

(2.) Suppose that \( f : [-a,a] \to \mathbb{R} \) is integrable and odd \( (f(x) = -f(x) \text{ for all } x \in [-a,a]). \) Show that \( \int_{-a}^{a} f(x) \, dx = 0 \).

One solution is to use Riemann’s point sum to approximate the integral. Since \( f \) is integrable, if \( I = \int_{-a}^{a} f(x) \, dx \) then \( S(f,P,T) \to I \) as \( \|P\| \to 0 \) (see problem 10). This means that for every \( \varepsilon > 0 \) there is a partition \( P_\varepsilon \) such that for any refinement \( Q \supseteq P_\varepsilon \) where \( Q = \{-a = x_0 < x_1 < \cdots < x_n = a\} \) and for any choice of sampling points \( T = (t_1, t_2, \ldots, t_n) \) where \( t_i \in (x_{i-1}, x_i) \), we have that the Riemann Point Sum satisfies \( |S(f,Q,T) - I| < \varepsilon \).
Choose any \( \varepsilon > 0 \) and let \( P_\varepsilon \) be the corresponding partition. Throw in the reflections of all division points of \( P_\varepsilon \) and zero to get a refined symmetric partition \( Q = P_\varepsilon \cup (-P_\varepsilon) \cup \{0\} \), where \( -P_\varepsilon = \{-t : t \in P_\varepsilon\} \). (So \( Q = -Q \).) Hence we may use an unusual numbering with \( 2n \) intervals for \( Q \) so that \( x_{-i} = -x_i \),

\[
Q = \{-a = x_{-n} < x_{1-n} < \cdots < x_{-1} < x_0 = 0 < x_1 < \cdots < x_n = a\}.
\]

We also choose symmetric sample points \( T = (t_{1-n}, t_{2-n}, \ldots, t_n) \) where \( t_i \in (x_{i-1}, x_i) \) and \( t_{1-j} = -t_j \in (x_{-j}, x_{1-j}) \) for \( i, j = 1, \ldots, n \). Now since \( f \) is odd, this implies that \( f(t_{1-k}) = -f(t_k) \) for all \( k \). Thus after changing dummy index \( j = 1 - k \), the Riemann sum is

\[
S(f, Q, T) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})
\]

\[
= \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) + \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})
\]

\[
= \sum_{k=1}^{n} f(t_{1-k})(x_{1-k} - x_{-k}) + \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})
\]

\[
= \sum_{k=1}^{n} f(t_k)(-x_{k-1} + x_k) + \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})
\]

\[
= 0.
\]

Thus we have shown that for all \( \varepsilon > 0 \), \( |0 - I| = |S(f, Q, T) - I| < \varepsilon \) so \( I = 0 \).

Another method is to split the integral as the sum \( I = \int_{-a}^{0} f(x) \, dx + \int_{0}^{a} f(x) \, dx \) and then change variables \( x = \varphi(\xi) = -\xi \) in the first integral \( \int_{-a}^{0} f(x) \, dx = \int_{a}^{0} f(\varphi(\xi)) \varphi'(\xi) \, d\xi = -\int_{0}^{a} f(-\xi) \, d\xi = -\int_{0}^{a} f(\xi) \, d\xi \) which cancels the second integral.

(3.) Suppose \( a < b \), \( 0 < k \) and \( f : [a, b] \to \mathbb{R} \) is an integrable function such that \( f(x) > k \) for all \( x \in [a, b] \). Show that \( h(x) = \frac{1}{f(x)} \) is integrable on \([a, b]\).

We are to show that \( \frac{1}{f} \) is bounded and for every \( \varepsilon > 0 \) there is a partition \( P \) such that the upper sum minus the lower sum \( U(h, P) - L(h, P) < \varepsilon \).

As \( f \) is integrable, it is bounded: there is \( M \in \mathbb{R} \) so that \( k \leq f(x) \leq M \) for all \( x \in [a, b] \). But as \( k > 0 \) we conclude that \( \frac{1}{M} \leq \frac{1}{f(x)} \leq \frac{1}{k} \) for all \( x \in [a, b] \), thus \( \frac{1}{f} \) is bounded.

Choose \( \varepsilon > 0 \). As \( f \) is integrable, there is a partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) so that \( U(f, P) - L(f, P) < k^2 \varepsilon \). Consider the sup and inf over each of the subintervals

\[
m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).
\]

Since \( k \leq m_i \leq f(x) \leq M_i \) for \( x \in [x_{i-1}, x_i] \), sup and inf for \( \frac{1}{f} \) satisfy on this interval

\[
\frac{1}{M_i} \leq \frac{1}{f(x)} \leq \frac{1}{m_i} \implies \frac{1}{M_i} \leq m'_i = \inf_{x \in [x_{i-1}, x_i]} \frac{1}{f(x)}, \quad M'_i = \sup_{x \in [x_{i-1}, x_i]} \frac{1}{f(x)} \leq \frac{1}{m_i}.
\]
Thus estimating the upper and lower sums for $\frac{1}{f}$, since all $(x_i - x_{i-1}) \geq 0$,

$$U \left( \frac{1}{f}, P \right) - L \left( \frac{1}{f}, P \right) = \sum_{j=1}^{n} (M_i' - m_i')(x_i - x_{i-1})$$

$$\leq \sum_{j=1}^{n} \left( \frac{1}{m_i} - \frac{1}{M_i} \right) (x_i - x_{i-1})$$

$$= \sum_{j=1}^{n} \frac{(M_i - m_i)}{m_i M_i} (x_i - x_{i-1})$$

$$\leq \sum_{j=1}^{n} \frac{(M_i - m_i)}{k^2} (x_i - x_{i-1})$$

$$= \frac{1}{k^2} (U(f, P) - L(f, P)) < \frac{k^2 \varepsilon}{k^2} = \varepsilon. $$

(4.) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions such that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \neq 0$. Suppose that $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \pi$. Show that $\lim_{x \to 0} \frac{f(x)}{g(x)} = \pi$.

This is just a particular case of l'Hôpital's Rule. So we may follow Proof (4.18). First notice that $g(x) - g(0) \neq 0$ and $g'(x) \neq 0$ if $x \neq 0$, so we may divide by them. By the sequential characterization of limits, it suffices to show that $\lim_{k \to \infty} \frac{f(x_i)}{g(x_i)} = \pi$ for every sequence $\{x_i\} \subset \mathbb{N}$ such that $x_i \neq 0$ for all $i$ and $x_i \to 0$ as $i \to \infty$. Similarly by the equivalence of the existence of a limit to the existence of equal left and right limits, we may assume $x_i > 0$ for all $i$ or $x_i < 0$ for all $i$. As $f$ and $g$ are continuous at 0, we have $f(0) = g(0) = 0$. We may suppose that $0 < x_i$ for all $i$. Now, since $f$ is assumed to be continuous on $[0, x_i]$ and differentiable on $(0, x_i)$ (actually we assumed more,) by the Generalized Mean Value Theorem there is a $c_i \in (0, x_i)$ so that

$$\frac{f(x_i)}{g(x_i)} = \frac{f(x_i) - f(0)}{g(x_i) - g(0)} = \frac{f'(c_i)}{g'(c_i)} \to \pi \quad \text{as } i \to \infty.$$ 

Now, letting $i \to \infty$, we have $x_i \to 0$. Since $0 < c_i < x_i$, by the Squeeze Theorem, $c_i \to 0$, so the conclusion follows since the sequential characterization implies $\lim_{i \to \infty} \frac{f'(c_i)}{g'(c_i)} = \pi$. A similar argument on the left side gives the same conclusion.

(5.) Let $f$ be continuous on $[a, b]$ and that $\int_{a}^{x} f(t) \, dt = \int_{a}^{b} f(t) \, dt$ for all $x \in [a, b]$. Then $f(x) = 0$ for all $x \in [a, b]$.

Observe that if $x \in [a, b]$ then

$$\int_{a}^{x} f(t) \, dt = \int_{x}^{b} f(t) \, dt = \left. \int_{a}^{b} f(t) \, dt - \int_{a}^{x} f(t) \, dt \right) \implies \int_{a}^{x} f(t) \, dt = \frac{1}{2} \int_{a}^{b} f(t) \, dt = \text{const.}$$

Thus, since $f$ is continuous, by the Fundamental Theorem of Calculus, for all $x \in [a, b]$, the primitive function is differentiable and equals

$$f(x) = \frac{d}{dx} \int_{a}^{x} f(t) \, dt = \frac{d}{dx} \text{const.} = 0.$$ 

(6.) Let $f(x) = x + x^3 + x^5$. Show $f$ has a continuous inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$. Find $(f^{-1})'(3)$ if possible.

First observe that since $f$ is a polynomial, it is differentiable on $\mathbb{R}$. Second, observe that $f$ is strictly increasing on $\mathbb{R}$. (This is like Theorem 4.24.) To see this, choose any numbers $a < b$ and since $f$ is
continuous on \([a, b]\) and differentiable on \((a, b)\), by the Mean Value Theorem there is a \(c \in (a, b)\) so that \(f(b) - f(a) = f'(c)(b - a)\). However \(f'(c) = 1 + 3c^2 + 5c^4 > 0\), so this implies \(f(b) > f(a)\). Third we apply the Inverse Function Theorem 4.26. As \(f\) is strictly increasing (so one to one,) and continuous on \(\mathbb{R}\), there is a continuous, strictly increasing inverse function on \(f(\mathbb{R}) = \mathbb{R}\) (since \(f\) is not bounded above and not bounded below.) By Theorem 4.27, since \(f\) is one to one, continuous and differentiable at \(x_0\) where \(f'(x_0) = 1 + 3x_0^2 + 5x_0^4 \neq 0\), the inverse function is differentiable. If \(f(x_0) = y_0\) (for example \(f(1) = 3\)) then

\[
(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \implies (f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{1 + 3(1^2) + 5(1^4)} = \frac{1}{9}.
\]

(7.) Suppose that \(x > -1\). Show that \(-\frac{1}{\sqrt{1 + x}} \geq 1 - \frac{x}{2}\).

This is an application of the Mean Value Theorem. Let \(h(x) = (1 + x)^{-1/2} - 1 + \frac{1}{2}x\). This function is differentiable for \(x > -1\). Also notice that \(h(0) = 0\). Then for any \(x > 0\) there is a \(c \in (0, x)\) so that \(h(x) = h(x) - h(0) = h'(c)(x - 0)\). But \(h'(c) = \frac{1}{2}(1 - (1 + c)^{-3/2}) > 0\) for \(c > 0\) because \((1 + c)^{-3/2} < 1\). Hence \(h(x) > 0\). Similarly, if \(-1 < x < 0\) then there is a \(c \in (x, 0)\) so that \(-h(x) = h(0) - h(x) = h'(c)(0 - x)\). This time \(h'(c) < 0\) because \((1 + c)^{-3/2} > 0\). Hence also \(h(x) > 0\). Putting inequalities on the two intervals and at zero together, \(h(x) \geq 0\) for all \(x > -1\).

(8.) Calculate \(\lim_{h \to 0} \frac{1}{h} \int_{-3}^{-3+h} e^{t^2} \, dt\).

Let \(F(x) = \int_0^x e^{t^2} \, dt\). Since \(f(t) = e^{t^2}\) is continuous as it is the composition of continuous functions, \(F(x)\) is differentiable and \(F' = f\), by the Fundamental Theorem of Calculus. The limit becomes the limit of a difference quotient

\[
\lim_{h \to 0} \frac{1}{h} \int_{-3}^{-3+h} e^{t^2} \, dt = \lim_{h \to 0} \frac{F(3 + h) - F(3)}{h} = F'(3) = f(3) = e^9.
\]

(9.) Suppose \(g : \mathbb{R} \to \mathbb{R}\) is a continuous function. Find \(\frac{d}{dt} \int_0^t g(x - t) \, dx\).

We do not yet have the machinery to pass derivatives through integrals. Therefore we shall change variables in the integral to put the dependence on \(t\) in the limits of integration. To that end, choose \(t, x_0\) so that \(x_0 < -|t|\). Let \(F(t) = \int_{x_0}^0 g(\xi) \, d\xi\). Since \(t\) is constant as far as the integration is concerned, change variables according to \(x = \varphi(\xi) = \xi + t\). Then since \(\varphi\) is continuously differentiable on \(\mathbb{R}\), and \(g\) is continuous on \(\varphi(\mathbb{R}) = \mathbb{R}\), we have by the change of variables formula,

\[
\int_0^t g(x - t) \, dx = \int_{t \varphi(0)}^{t \varphi(t)} g(x - t) \, dx = \int_{-t}^0 g(\varphi(\xi) - t) \varphi'(\xi) \, d\xi = \int_{-t}^0 g(\xi) \cdot 1 \, d\xi = F(0) - F(0 - t).
\]

Since \(g\) is continuous on \(\mathbb{R}\), by the Fundamental Theorem of Calculus, \(F\) is differentiable and \(F'(\xi) = g(\xi)\). Now, by the chain rule,

\[
\frac{d}{dt} \int_0^t g(x - t) \, dx = \frac{d}{dt} (F(0) - F(-t)) = 0 - F'(0 - t) \cdot (-1) = g(0).
\]

(10.) Let \(P = \{a = x_0 < x_1 < \cdots < x_n = b\}\) be a partition of \([a, b]\). The mesh \(\|P\| = \max\{x_i - x_{i-1} : i = 1, \ldots, n\}\) is the length of the largest subinterval. Let \(f : [a, b] \to \mathbb{R}\) be a bounded function. Show that \(f\) is integrable on \([a, b]\) if and only if

\(\text{ (i) For all } \varepsilon > 0 \text{ there is } \delta > 0 \text{ so that } U(f, P) - L(f, P) < \varepsilon \text{ whenever } \|P\| < \delta. \)

You may use the theorem that \(f\) is integrable on \([a, b]\) if and only if
For all $\varepsilon > 0$ there is a partition $P$ so that $U(f, P) - L(f, P) < \varepsilon$.

Suppose $\{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$ such that $U(f, Q) - L(f, Q) < \frac{\varepsilon}{2}$ for some partition $Q = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and $\delta = \frac{\varepsilon}{2} > 0$. Let $\delta = \frac{\varepsilon}{2K}$, then the subintervals $[x_i, x_{i+1}]$ are $\frac{\varepsilon}{2}$. Thus by uniform convergence, for any partition $P$ of $[a, b]$ such that $\|P\| < \frac{\varepsilon}{2}$, no two subintervals $[x_i, x_{i+1}]$ are $\frac{\varepsilon}{2}$.
Let $M'_i = \sup_{[x_{i-1}, x_i]} f$ and $m'_i = \inf_{[x_{i-1}, x_i]} f$. Thus $M'_i \leq M_i + \frac{\varepsilon}{5(b-a)}$ and $m'_i \geq m_i - \frac{\varepsilon}{5(b-a)}$. Estimating,

$$U(f, P) - L(f, P) = \sum_{i=1}^{m} (M'_i - m'_i) (x_i - x_{i-1}) \leq \sum_{i=1}^{m} \left( M_i + \frac{\varepsilon}{5(b-a)} - m_i + \frac{\varepsilon}{5(b-a)} \right) (x_i - x_{i-1}) = U(f_n, P) - L(f_n, P) + \frac{2\varepsilon}{5(b-a)} \sum_{i=1}^{m} (x_i - x_{i-1}) \leq \frac{\varepsilon}{2} + \frac{2\varepsilon(b-a)}{5(b-a)} < \varepsilon.$$

So $f$ is integrable on $[a, b]$. By uniform convergence, for all $n > R$ and all $x \in [a, b]$, $|f_n(x) - f(x)| < \frac{\varepsilon}{5(b-a)}$ implies

$$\left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right| \leq \int_a^b |f_n(x) - f(x)| \, dx \leq \int_a^b \frac{\varepsilon}{5(b-a)} \, dx = \frac{\varepsilon(b-a)}{5(b-a)} < \varepsilon.$$

As $\varepsilon$ was arbitrary, this implies (K).

(12.) Let $p > 0$. Show that the improper integral of $f(x) = \frac{\cos x}{(\ln x)^p}$ exists on $(e, \infty)$.

Since $f$ is continuous, on each interval $[e, R]$ the function $f$ is integrable. The improper integral $\int_e^R f(x) \, dx$ exists provided $\lim_{R \to \infty} \int_e^R f(x) \, dx$ exists as a finite number.

The idea is to split the integral $I(R) = \int_e^R f(x) \, dx$ into three parts, each of which converges as $R \to \infty$. Let $\alpha = \frac{7\pi}{2} > e$, $\beta(R) = 2 \pi k(R) + \frac{3\pi}{2}$ where $k(R) \in \mathbb{N}$ such that $2 \pi k(R) + \frac{3\pi}{2} \leq R < 2 \pi k(R) + \frac{7\pi}{2}$ for $R > \frac{11\pi}{2}$, $I_1 = \int_e^\alpha f(x) \, dx$, $I_2 = \int_{\beta(R)}^{\alpha} f(x) \, dx$ and $I_3 = \int_{\beta(R)}^{R} f(x) \, dx$. Then $I(R) = I_1 + I_2 + I_3$. $I_1$ is constant for all $R$. Since $(\ln x)^p$ is increasing for $x \geq e$ and tends to infinity as $x \to \infty$, and $\beta(R) \to \infty$ as $R \to \infty$,

$$|I_3(R)| \leq \int_{\beta(R)}^{R} \left| \frac{\cos x}{(\ln x)^p} \right| \, dx \leq \int_{\beta(R)}^{R} \frac{\, dx}{(\ln \beta(R))^p} = \frac{R - \beta(R)}{(\ln \beta(R))^p} \leq \frac{2\pi}{(\ln \beta(R))^p} \to 0 \text{ as } R \to \infty.$$

Finally, since $\cos(x) = -\cos(x + \pi)$,

$$I_2 = \sum_{j=2}^{k(R)+1} \int_{2\pi j - \frac{3\pi}{2}}^{2\pi j + \frac{3\pi}{2}} \frac{\cos x \, dx}{(\ln x)^p} = \sum_{j=2}^{k(R)+1} \int_{2\pi j - \frac{3\pi}{2}}^{2\pi j + \frac{3\pi}{2}} \left( \frac{1}{[\ln(2\pi j + \pi + x)]^p} - \frac{1}{[\ln(2\pi j + \pi - x)]^p} \right) \cos x \, dx.$$

$I_2(k(R))$ converges as $R \to \infty$ if and only if $I_2(k)$ converges as $k \to \infty$. But the parenthesis term and $\cos x$ are both positive, so that $I_2(k)$ is an increasing sequence in $k$, and hence converges if and only if it is bounded. But using $\cos x \leq 1$ and $(\ln s)^p$ is increasing,

$$I_2 \leq \pi \sum_{j=2}^{k(R)+1} \left( \frac{1}{[\ln(2\pi j - \frac{3\pi}{2})]^p} - \frac{1}{[\ln(2\pi j + \frac{3\pi}{2})]^p} \right) = \pi \left( \frac{\pi}{[\ln(\frac{\pi}{2})]^p} - \frac{\pi}{[\ln(\frac{15\pi}{2})]^p} \right) \leq \frac{\pi}{[\ln(\frac{7\pi}{2})]^p}$$

since the sum telescopes.
(13.) Evaluate the improper integral \( I = \int_{0}^{\pi/2} \frac{\cos x}{(\sin x)^{1/3}} \, dx \), if it exists.

The function is continuous on \((0, \pi/2]\) so it is integrable on every interval \([\varepsilon, \pi/2]\) where \(0 < \varepsilon < \pi/2\). Thus the improper integral exists if \( \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{\pi/2} \frac{\cos x}{(\sin x)^{1/3}} \, dx \) has a finite limit. But changing variables \( u = \varphi(x) = \sin x \),

\[
\int_{\varepsilon}^{\pi/2} \frac{\cos x}{(\sin x)^{1/3}} \, dx = \int_{\varepsilon}^{\pi/2} \frac{\varphi'(x)}{(\varphi(x))^{1/3}} \, dx = \int_{\sin \varepsilon}^{1} \frac{du}{u^{1/3}} = \left. \frac{3}{2} u^{2/3} \right|_{u=\sin \varepsilon}^{1} = \frac{3}{2} \left( 1^{2/3} - (\sin \varepsilon)^{2/3} \right) - \frac{3}{2}
\]

as \( \varepsilon \to 0^+ \), so the improper integral exists and \( I = 3/2 \).