1. Let $f : \mathbb{R} \to \mathbb{R}$.

(a.) [5] State the definition: $f$ is differentiable at $a$.

(b.) [16] Determine whether the function is differentiable at 0. Justify your answer.

$$f(x) = \begin{cases} 
\frac{x^2}{\sqrt{x^2 + x^4}} & \text{if } x \neq 0; \\
0 & \text{if } x = 0.
\end{cases}$$
2. Suppose \( f : [0, 2\pi] \rightarrow \mathbb{R} \) is a continuous function and that \( f(q) = 0 \) for every rational number \( q \in [0, 2\pi] \cap \mathbb{Q} \). Show that \( f(x) = 0 \) for all \( x \in [0, 2\pi] \).
3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) [7] Let \((\mathcal{F}, +, \cdot, 0, 1)\) be a field. If \(x, y \in \mathcal{F}\) such that \(x \neq 0\) satisfy \(x \cdot y = x\) then \(y = 1\).

   \[\begin{array}{c}
   \text{TRUE: } \square \\
   \text{FALSE: } \square 
   \end{array}\]

(b) [7] The sequence \(\left\{ \frac{n - 1}{n} \right\}\) is a Cauchy sequence.

   \[\begin{array}{c}
   \text{TRUE: } \square \\
   \text{FALSE: } \square 
   \end{array}\]

(c) [7] If \(f_n, g : \mathbb{R} \to \mathbb{R}\) are functions such that \(|f_n(x) - g(x)| < \frac{1}{2^n}\) for all \(x \in \mathbb{R}\) and for all \(n \in \mathbb{N}\). Then \(f_n \to g\) uniformly in \(\mathbb{R}\).

   \[\begin{array}{c}
   \text{TRUE: } \square \\
   \text{FALSE: } \square 
   \end{array}\]
4. Let \( E = \left\{ \int_0^\pi f(x) \sin x \, dx \bigg| f : [0, \pi] \to \mathbb{R} \text{ is continuous and } f(x) > 0 \text{ for all } x \in [0, \pi] \right\} \).

(a) [5] Show that \( E \) is nonempty and bounded below.

(b) [16] What is the greatest lower bound of \( E \)? Does the set \( E \) have a minimum? Justify your answers.
5. [21] Prove the following theorem:

**Theorem.** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and $f'(x)$ is bounded on $\mathbb{R}$, then $f$ is uniformly continuous on $\mathbb{R}$. 
6. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) [7] If \( f : [0, \infty) \to \mathbb{R} \) is continuous, positive and \( f(x) \to 0 \) as \( x \to \infty \) then the improper integral \( \int_0^\infty f(x) \, dx \) converges.

\[ \begin{array}{ll}
\text{TRUE: } & \bigcirc \\
\text{FALSE: } & \bigcirc \\
\end{array} \]

(b) [7] If \( \sum_{k=1}^\infty a_k \) and \( \sum_{k=1}^\infty b_k \) are convergent series then \( \sum_{k=1}^\infty (a_k + b_k) \) is a convergent series.

\[ \begin{array}{ll}
\text{TRUE: } & \bigcirc \\
\text{FALSE: } & \bigcirc \\
\end{array} \]

(c) [8] If \( f : [0, 1] \to \mathbb{R} \) is nonnegative and bounded, then it is integrable on \([0, 1] \).

\[ \begin{array}{ll}
\text{TRUE: } & \bigcirc \\
\text{FALSE: } & \bigcirc \\
\end{array} \]
7. [22] Let $a < b$ and $f : [a, b] \to \mathbb{R}$. Show that if for all $\varepsilon > 0$ there are integrable functions $g, h : [a, b] \to \mathbb{R}$ such that $g(x) \leq f(x) \leq h(x)$ for all $x \in [a, b]$ and $\int_a^b h(x) - g(x) \, dx < \varepsilon$, then $f$ is integrable on $[a, b]$. 
8. Determine whether the following series are absolutely convergent, conditionally convergent or divergent. In each case you must justify your answer.

(a) \[ \sum_{k=1}^{\infty} \frac{(-1)^k \log k}{k}, \quad \text{Abs. Conv.: \(\bigcirc\), Cond. Conv.: \(\bigcirc\), Divergent: \(\bigcirc\)} \]

(b) \[ \sum_{k=1}^{\infty} \frac{(-1)^k \log k}{\log(k^2 + k + 1)}, \quad \text{Abs. Conv.: \(\bigcirc\), Cond. Conv.: \(\bigcirc\), Divergent: \(\bigcirc\)} \]

(c) \[ \sum_{k=1}^{\infty} \frac{(-1)^k \log k}{k^2}, \quad \text{Abs. Conv.: \(\bigcirc\), Cond. Conv.: \(\bigcirc\), Divergent: \(\bigcirc\)} \]
9. [22] Prove the following theorem:

**Theorem.** Suppose that $\sum_{k=1}^{\infty} a_k$ is an absolutely convergent series and that $\{b_k\}$ is a bounded sequence. Then $\sum_{k=1}^{\infty} a_k b_k$ is an absolutely convergent series.