Math 3210 § 2.	**Final Exam**	Name:__________________
Treibergs		December 16, 2009

You are allowed "cheat sheets," two 8"×11.5" pages of notes. Otherwise this is a closed book test. No other books, papers, calculators, laptops or messaging devices are permitted. Give complete solutions. Be clear about the order of logic and state the theorems and definitions that you use. There are [150] total points. **Do SEVEN of nine problems.** If you do more than seven problems, only the first seven will be graded. Cross out the problems you don't wish to be graded.

1.	/21
2.	/21
3.	/21
4.	/21
5.	/21
6.	/22
7.	/22
8.	/22
9.	/22
Total	/150

1. Let $f : \mathbb{R} \to \mathbb{R}$.

(a.) [5] State the definition: f is differentiable at a.

(b.) [16] Determine whether the function is differentiable at 0. Justify your answer.

$$f(x) = \begin{cases} \dfrac{x^2}{\sqrt{x^2 + x^4}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Math 3210 § 2. **Final Exam** Name:____________________

Treibergs December 16, 2009

Your grades will be posted outside my office according to your *Secret I. D.:*

2. [21] Suppose $f : [0, 2\pi] \to \mathbb{R}$ is a continuous function and that $f(q) = 0$ for every rational number $q \in [0, 2\pi] \cap \mathbb{Q}$. Show that $f(x) = 0$ for all $x \in [0, 2\pi]$.

3. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) [7] Let $(\mathcal{F}, +, \cdot, 0, 1)$ be a field. If $x, y \in \mathcal{F}$ such that $x \neq 0$ satisfy $x \cdot y = x$ then $y = 1$.

TRUE: ◯	FALSE: ◯

(b) [7] The sequence $\left\{ \dfrac{n-1}{n} \right\}$ is a Cauchy sequence.

TRUE: ◯	FALSE: ◯

(c) [7] If $f_n, g : \mathbb{R} \to \mathbb{R}$ are functions such that $|f_n(x) - g(x)| < \dfrac{1}{2^n}$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Then $f_n \to g$ uniformly in $\mathbb{R}$.

TRUE: ◯	FALSE: ◯

4. Let $E = \left\{ \int_0^\pi f(x) \sin x \, dx \;\middle|\; f : [0.\pi] \to \mathbb{R} \text{ is continuous and } f(x) > 0 \text{ for all } x \in [0, \pi] \right\}.$

   (a) [5] Show that E is nonempty and bounded below.

   (b) [16] What is the greatest lower bound of E? Does the set E have a minimum? Justify your answers.

5. [21] Prove the following theorem:

   Theorem. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and $f'(x)$ is bounded on $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.*

6. Determine whether the following statements are true or false. If true, give a proof. If false, give a counterexample.

   (a) [7] If $f : [0, \infty) \to \mathbb{R}$ is continuous, positive and $f(x) \to 0$ as $x \to \infty$ then the improper integral $\displaystyle\int_0^\infty f(x)\,dx$ converges.

TRUE: ◯	FALSE: ◯

   (b) [7] If $\displaystyle\sum_{k=1}^\infty a_k$ and $\displaystyle\sum_{k=1}^\infty b_k$ are convergent series then $\displaystyle\sum_{k=1}^\infty (a_k + b_k)$ is a convergent series.

TRUE: ◯	FALSE: ◯

   (c) [8] If $f : [0, 1] \to \mathbb{R}$ is nonnegative and bounded, then it is integrable on $[0, 1]$.

TRUE: ◯	FALSE: ◯

7. [22] Let $a < b$ and $f : [a, b] \to \mathbb{R}$. Show that if for all $\varepsilon > 0$ there are integrable functions $g, h : [a, b] \to \mathbb{R}$ such that $g(x) \le f(x) \le h(x)$ for all $x \in [a, b]$ and $\displaystyle\int_a^b h(x) - g(x)\, dx < \varepsilon$, then f is integrable on $[a, b]$.

8. Determine whether the following series are absolutely convergent, conditionally convergent or divergent. In each case you must justify your answer.

   (a) [7] $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\log k}{k}$, Abs. Conv.: ◯ Cond. Conv.: ◯ Divergent: ◯

   (b) [7] $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\log k}{\log(k^2+k+1)}$, Abs. Conv.: ◯ Cond. Conv.: ◯ Divergent: ◯

   (c) [8] $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\log k}{k^2}$, Abs. Conv.: ◯ Cond. Conv.: ◯ Divergent: ◯

9. [22] Prove the following theorem:

Theorem. *Suppose that* $\sum_{k=1}^{\infty} a_k$ *is an absolutly convergent series and that* $\{b_k\}$ *is a bounded sequence. Then* $\sum_{k=1}^{\infty} a_k b_k$ *is an absolutely convergent series.*