Homework for Math 3010 §1, Spring 2018

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Our text is by John Stillwell, *Mathematics and its History*, 3rd. ed., Springer, New York, 2010. Please read the relevant sections in the text as well as any cited reference. Assignments are due the following Friday, or on April 24, whichever comes first.

Your written work reflects your professionalism. Make answers complete, self contained and written in good English. This means that you should copy or paraphrase each question, provide adequate explanation to help the reader understand the structure of your argument, be thorough in the details, state any theorem that you use and proofread your answer.

Homework from Wednesday to Monday will be due Friday. Late homework that is up to one week late will receive half credit. Homework that is more than one week late will receive no credit at all. The homework reader is Lisa Penfold. Homework that is placed in her mailbox in JWB 228 before she picks it up not later than 5:00 pm Friday afternoon will be considered to be on time.

Please hand in problems A1 on Friday, January 12.

A1. Please hand in the following exercises from from Stillwell’s *Mathematics and its History*.

1.2.3 Show that any integer square leaves remainder 0 or 1 on division by 4.

1.2.4 Deduce from [1.2.3] that if $(a, b, c)$ is any Pythagorean triple then $a$ and $b$ cannot both be odd.

1.3.1 Deduce that if $(a, b, c)$ is any Pythagorean triple then

$$\frac{a}{c} = \frac{p^2 - q^2}{p^2 + q^2}, \quad \frac{b}{c} = \frac{2pq}{p^2 + q^2},$$

for some integers $p, q$. [Hint: This question is asks you to give an argument that shows Pythagorean Triples are given by the formula in the middle of page 4.]

1.4.1 What has this figure to do with the Pythagorean Theorem?
1.4.2 In Book VI of *Elements*, Euclid gives the following argument for the Pythagorean Theorem based on similar triangles. Show that the three triangles in the figure are similar, and hence prove the Pythagorean theorem by equating ratios of corresponding sides.

Please hand in problems B1–B3 on Friday, January 19.

B1. Please hand in the following exercises from Stillwell’s *Mathematics and its History*.

2.2.1 Show that for both cube and octohedron,

\[
\frac{\text{circumradius}}{\text{inradius}} = \sqrt{3}
\]

2.2.2 Check Pacioli’s construction of the regular isosahedron. Consider the rectangle \([-\phi, \phi] \times \([-1, 1]\). Suppose that three such rectangles are centered in the coordinate planes oriented so that the long directions align with the three axes as in the figure. Show that \(AB = AC = BC\). Recall that \(\phi = \frac{1}{2} + \frac{\sqrt{5}}{2}\) satisfies \(\phi^2 = 1 + \phi\).

B2. Show that the Golden section \(\phi\) is irrational.
B3. Check Euclid’s construction of the regular pentagon. Suppose that collinear points $A$, $B$, $C$ have distances $a = AB$, $\phi a = AC$. Construct the isosceles triangle $\triangle(BDC)$ with $a = BD = CD$. Let $M$ be the bisector between $B$ and $C$. Let the angle $\alpha = \angle(ACD)$.

\[\begin{align*}
\text{a. Find the length of } DM \text{ using the triangle } \triangle(BMD) \\
\text{b. Show that } AD = AC \text{ using the triangle } \triangle(AMD). \\
\text{c. Express the angles of triangle } \triangle(ABD) \text{ in terms of } \alpha \text{ and show that } \alpha = 72^\circ. \text{ Thus it is the central angle of a sector of a regular pentagon. HINT: The sum of the interior angles of any triangle is } 180^\circ.
\end{align*}\]

Please hand in problems C1–C3 on Friday, January 26.

C1. Please hand in the following exercises from Stillwell’s *Mathematics and its History*.

2.5.1 Using $X$ and $Y$ for the horizontal and vertical components, show that the straight line $RP$ in Figure 2.10 of the text has equation

\[Y = \frac{\sqrt{1 - x^2}}{1 + x}(X - 1).\]

2.5.2 Deduce the equation of the Cissoid of Diocles from Exercise 2.5.1.
B2. Given a line in the plane and a point \( P \) not on the line, describe how to construct a new line through \( P \) which is parallel to the first line by using straightedge and compass. Explain why your construction does the job.

B3. Here is how Menaechmus constructed the parabola as the locus of points \( P = \{(x, y)\} \). Given points \( A, O \) and \( X \) on line such that \( a \) is the distance \( AO \) and \( x \) is the distance \( OX \). Let \( L \) be a perpendicular line through \( O \). For each \( x \), construct a circle whose diameter is \( AX \). Then \( y \) is the distance from \( O \) to the intersection point of \( L \) with the circle. Show that Menaechmus’ construction yields the parabola. Show also that the parabola passes through the point \((a, a)\).

D1. Using the Euclidean algorithm, find \( \gcd(963, 657) \).

D2. Find integers \( x \) and \( y \) that satisfy

\[
\gcd(24, 138) = 24x + 138y.
\]

D3. Show that if \( a \mid bc \) and \( \gcd(a, b) = 1 \) then \( a \mid c \).

D4. Please hand in the following exercises from Stillwell’s *Mathematics and its History*.

4.3.2 Similarly to [4.3.1], show that any two tetrahedra with the same base and height can be approximated arbitrarily closely by the same prisms, differently stacked, as in Figure 4.6 of the text.

4.3.3 Deduce from exercise 4.3.2 that two tetrahedra with the same base and height have equal volume.
If you have not had Math 3210, you may assume that the apex of the tetrahedron is directly over the base, which yields to a simpler analysis. Be careful on 4.3.2 and 4.3.3. You have to take into account how skew your tetrahedron is. If $T$ and $T'$ are the tetrahedra and $S_n$ and $S'_n$ are the approximating stacks of prisms with $n$ levels, we have volumes $V(S_n) = V(S'_n)$ but we need to establish for Eudoxus’s exhaustion method, that for any positive number $\varepsilon > 0$, the stacks and tetrahedra are close enough so that both $|V(T) - V(S_n)| < \varepsilon$ and $|V(T') - V(S'_n)| < \varepsilon$ for some $n$ large enough, depending on $\varepsilon$. You will need to compare inner and outer approximations for this. A problem occurs if the stacks spill outside the tetrahedron, as in the second diagram.

We conclude from the triangle inequality, that for any $\varepsilon > 0$ there is an $n$ such that

$$|V(T) - V(T')| = |V(T) - V(S_n) + V(S_n) - V(S'_n) + V(S'_n) - V(T')|$$

$$\leq |V(T) - V(S_n)| + |V(S_n) - V(S'_n)| + |V(S'_n) - V(T')|$$

$$< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon.$$ 

Since $\varepsilon$ may be any positive number no matter how small, $V(T) = V(T')$.

**D5.** On a separate piece of paper, write your Essay on Mathematics of Antiquity proposal. After the proposal is returned to you, please hand your proposal in again when you hand in your essay next week. Be sure to include in your proposal

- Working Title
- Short but specific description of what your essay is about. Don’t just say you will discuss what the Greeks thought about $\pi$. Better say that you will describe how Archimedes showed that $3 \frac{10}{71} < \pi < 3 \frac{1}{7}$. Everyone in class should have a different topic.
- State an interesting fact you’ve discovered about your topic in your preliminary readings.
- State which style manual you’ll follow. You can find a list at the Mariott website [http://campusguides.lib.utah.edu/style](http://campusguides.lib.utah.edu/style)
- Give two internet references. Please include the author and the URL.
- Give two book or journal references specific to your topic (other than Stillwell).

Please hand in your essay E1 on Friday, February 9.

**E1.** Essay on the Mathematics of Antiquity. Write an essay about a specific mathematical discovery/ theorem/ method that occurred before Christ.

- The paper should be five pages (in some reasonable font and font size) double-spaced and printed out on paper. It should be written in good technical English. It should be written for an audience of Math 3010 students.
- There must be some mathematics, and mathematical explanation, in your paper. Just how you incorporate some mathematical exposition will vary from subject to subject. Include displayed equations and diagrams if appropriate.
- You must draw on a bare minimum of three book and journal sources. It is good if you include “primary” sources, quoting directly from the mathematician you’re discussing, or at least from sources closest to them reconstructing the original source. A “Secondary” source is a scholarly interpretation later than the original subject of study in a book or journal. You may use blogs and Wiki articles provided that you give them credit. But also track down the source cited in a Wikipedia article.
• Give credit where it is due: whenever you use another author’s ideas, whether appearing in your paper as direct quotation, paraphrase, or simply influence, you must cite them (with a footnote and then include in the bibliography). Formatting these citations and bibliography entries should be unambiguous, according to your chosen style guide. (Parts of these instructions are quoted from Patrikis’s assignment 2-19-16.)

• Please attach your essay proposal from last week to your paper.

Please hand in problems F1–F6 on Friday, February 16.

F1. Find the least common multiple of 2646 and 8232.

F2. Use the Chinese square root algorithm to find \( \sqrt{142,884} \).

F3. Use the Chinese cube root algorithm to find \( \sqrt[3]{12,812,904} \).

F4. Solve Problem 1 from Chapter VII of Nine Chapters. Several people purchase in common one item. If each person paid 8 coins, the surplus is 3. If each paid 7, the deficiency is 4. How many people were there and what is the price of the item? [Katz, A History of Mathematics, 2009, p. 227]

F5. Solve Problem 3 from Chapter VIII of Nine Chapters using the Chinese method. None of the yields of two bundles of best grain, three bundles of ordinary grain and 4 bundles of the worst grain are sufficient to make a whole measure. If we add to the good bundles one bundle of the ordinary, to the ordinary bundles one bundle of the worst, and to the worst bundles one bundle of the best, then each yield is exactly one measure. How many measures does one bundle of each of the three types of grain contain? Show that the solution, according to the Chinese method involves the use of negative numbers. [Katz, p. 227]

F6. Solve Problem 20 from Chapter IX of Nine Chapters. A square walled city of unknown dimension has four gates at the centers of each side. A tree stands 20 pu north of the north gate. One must walk 14 pu south of the south gate and then turn west and walk 1775 pu before one can see the tree. What are the dimensions of the city? [Katz, p. 227]