MATH 2250—Midterm 1 Grading
Information and Solutions

- Graded midterms were handed back in class on Thursday, Feb. 14
- Each numbered question was worth ten points, for a possible total of 60 points.
- Class average was approximately 48.5/60
- There is also a letter grade on your graded midterm, which takes into account your midterm score and your first three homework scores.
- To get the letter grade, I first calculated a number grade (from 0 to 100) using the fact that the midterm will be 20% of the final grade and each homework will be 2.5% of the final grade:
  \[
  \text{NumberGrade} = \frac{H + M}{.275},
  \]
  where
  \[
  H = \left(\frac{\text{TotalHWPoints}}{90}\right)(.075) \quad \text{and} \quad M = \left(\frac{\text{MidtermPoints}}{60}\right)(.2)
  \]
- I rounded the number grade to one decimal place and assigned a letter grade as follows: A (95.0–100); A- (92.0–94.9); B+ (88.0–91.9); B (83.0–87.9); B- (80.0–82.9); C+ (74.0–79.9); C (65.0–73.9); C- (60.0–64.9); D+ (55.0–59.9); D (45.0–54.9); D- (40.0–44.9)
- The mean number grade was approximately 80, and the median number grade was approximately 83.
1. A population at time \( t \) (measured in years) is given by the function
\[ P(t) = 450,000e^{0.02t} \]

(a) Find the doubling time for the population. Hint: Use \( \ln 2 \approx 0.7 \)

SOLUTION:
Let \( T \) be the doubling time. Since \( P(0) = 450,000 \), it must be true that \( P(T) = 900,000 \) (doubled from time 0 to time \( T \)). Then,
\[ 900,000 = 450,000e^{0.02T}. \]

Solve for \( T \):
\[ 2 = e^{0.02T} \]
\[ \ln 2 = 0.02T \]
\[ T = \frac{\ln 2}{0.02} \approx 35 \text{ years}. \]

(b) Write down the differential equation and initial condition to which \( P(t) \) above is the solution.

SOLUTION:
\( P(t) \) above describes exponential growth, which we derived in class from a differential equation that assumes that the rate of increase of the population \( \left( \frac{dP}{dt} \right) \) is proportional to the size of the population \( P \), with proportionality constant \( k \). In this case, \( k = 0.02 \). The differential equation and initial condition are
\[ \frac{dP}{dt} = 0.02P, \quad P(0) = 450,000. \]
2. Solve the following Initial Value Problems for the function \( y(x) \).

(a) \( \frac{dy}{dx} = \frac{1}{x^2 + x} \), \( y(1) = 0 \)

SOLUTION:
The right-hand side depends only on \( x \), so we need only integrate and solve for the arbitrary constant:

\[
y(x) = \int \frac{1}{x^2 + x} \, dx
\]

We need to use a partial fractions expansion on the integrand, which leads to

\[
y(x) = \int \left[ \frac{1}{x} - \frac{1}{x+1} \right] \, dx
\]

\[
y(x) = \ln |x| - \ln |x + 1| + C
\]

The solution will be valid near the initial condition, where \( x = 1 \), so we may drop the absolute value symbols since \( x \) will be positive where the solution is valid. We also use the initial condition to solve for \( C \):

\[
y(1) = 0 - \ln 2 + C = 0, \quad \text{so } C = \ln 2
\]

Final answer:

\[
y(x) = \ln x - \ln(x + 1) + \ln 2
\]

(b) \( y'' = \sqrt{x} \), \( y(0) = 1 \quad y'(0) = 2 \)

SOLUTION:
We integrate twice and solve for the arbitrary constant each time:

\[
y' = \int x^{1/2} \, dx = \frac{2}{3} x^{3/2} + C_1
\]

\[
y'(0) = 0 + C_1 = 2, \quad \text{so } C_1 = 2, \quad \text{and } y' = \frac{2}{3} x^{3/2} + 2
\]

\[
y = \int \left[ \frac{2}{3} x^{3/2} + 2 \right] \, dx = \frac{4}{15} x^{5/2} + 2x + C_2
\]

\[
y(0) = 0 + 0 + C_2 = 1, \quad \text{so } C_2 = 1, \quad \text{and } y = \frac{4}{15} x^{5/2} + 2x + 1
\]
3. Find the general solution \( y(x) \) to the following differential equations.

(a) \( \frac{dy}{dx} = 3x^2e^y \)

SOLUTION:
Separate variables, integrate both sides, and solve for \( y \):

\[
\int e^{-y} \, dy = \int 3x^2 \, dx
\]

\[-e^{-y} = x^3 + C\]

\[e^{-y} = -x^3 - C\]

\[-y = \ln(-x^3 - C)\]

\[y = -\ln(-x^3 - C)\]

(b) \( 4xy^2 + y' = 5x^4y^2 \)

SOLUTION:
We can separate variables by noting that two of the terms have \( y^2 \) as the \( y \)-dependent factor:

\[y' = y^2(5x^4 - 4x)\]

\[
\int y^{-2} \, dy = \int [5x^4 - 4x] \, dx
\]

\[-y^{-1} = x^5 - 2x^2 + C\]

\[y^{-1} = -x^5 + 2x^2 - C\]

\[y = \frac{1}{-x^5 + 2x^2 - C}\]
4. Find the solution $y(x)$ to the Initial Value Problem

$$x \frac{dy}{dx} - 3y = x^3 + 2x, \quad y(1) = 0$$

**SOLUTION:**

First we need to put the equation into the form $\frac{dy}{dx} + P(x)y = Q(x)$:

$$\frac{dy}{dx} - \frac{3}{x}y = x^2 + 2$$

So $P(x) = -\frac{3}{x}$, and the integrating factor is

$$e^{\int -\frac{3}{x} \, dx} = e^{-3 \ln x} = e^{\ln x^{-3}} = x^{-3}$$

Multiply both sides by the integrating factor and proceed:

$$x^{-3} \frac{dy}{dx} - 3x^{-2}y = x^{-1} + 2x^{-3}$$

$$\frac{d}{dx} (x^{-3}y) = x^{-1} + 2x^{-3}$$

$$x^{-3}y = \int [x^{-1} + 2x^{-3}] \, dx$$

$$x^{-3}y = \ln x - x^{-2} + C$$

$$y = x^3 \ln x - x + Cx^3$$

Now use the initial condition to solve for $C$,

$$0 = 0 - 1 + C, \quad \text{so } C = 1$$

Finally,

$$y = x^3 \ln x - x + x^3$$
5. A 400-gal tank initially contains 100 gallons of brine containing 50 lb of salt. At time $t = 0$, brine containing 1 lb of salt per gallon starts entering the tank at the rate of 5 gallons/second, and the well-mixed brine in the tank flows out at the rate of 3 gallons/second; thus the tank is full at time $t = 150$ seconds. Find a formula for the amount of salt in the tank at time $t$ (for $0 \leq t \leq 150$).

SOLUTION:

Let $y(t)$ be the amount of salt (in lbs) in the tank at time $t$ (in seconds). Salt is entering the tank at a rate of 5 lbs/second (5 gallons/second times 1 lb/gallon). For salt leaving the tank, we know that the tank is draining brine at the rate of 3 gallons/second, and we know that there are $y$ lbs of salt in the tank at any time. We also need to know the gallons of brine that are in the tank at any time, which is changing as the tank fills. At time 0 there were 100 gallons, and the net increase is 2 gallons per second (5 in minus 3 out). So the volume of brine in the tank at time $t$ is $100 + 2t$. We can then write the initial value problem:

$$
\frac{dy}{dt} = 5 - \frac{3y}{100 + 2t}, \quad y(0) = 50
$$

This is a linear equation so we calculate an integrating factor:

$$
\rho(t) = e^{\int \frac{3}{100 + 2t} \, dt} = e^{\frac{3}{2} \ln(100 + 2t)} = e^{\ln((100 + 2t)^{3/2})} = (100 + 2t)^{3/2}
$$

$$
\frac{dy}{dt} \left(100 + 2t\right)^{3/2} + 3y(100 + 2t)^{1/2} = 5(100 + 2t)^{3/2}
$$

$$
\frac{d}{dt} \left[y(100 + 2t)^{3/2}\right] = 5(100 + 2t)^{3/2}
$$

$$
y(100 + 2t)^{3/2} = \int 5(100 + 2t)^{3/2} \, dt
$$

$$
y(100 + 2t)^{3/2} = \frac{2}{5} \frac{1}{2} (100 + 2t)^{5/2} + C
$$

$$
y(t) = 100 + 2t + C(100 + 2t)^{-3/2}
$$

$y(0) = 100 + 0 + C(100)^{-3/2} = 50$, so $C = -50(100)^{3/2} = -50000$

FINAL ANSWER:

$$
y(t) = 100 + 2t - \frac{50000}{(100 + 2t)^{3/2}}
$$
6. For the differential equation

$$\frac{dx}{dt} = (x + 1)(x - 1)(x - 3),$$

find the critical points (equilibria), determine whether each critical point is stable or unstable, and draw the phase diagram.

**SOLUTION:**

The critical points are the $x$ values at which $\frac{dx}{dt} = 0$, which are $x = -1$, $x = 1$, and $x = 3$. To determine stability, we can figure out the sign of $\frac{dx}{dt}$ in the four regions of the $x$-axis separated by the three critical points:

- For $x < -1$, $\frac{dx}{dt} < 0$
- For $-1 < x < 1$, $\frac{dx}{dt} > 0$
- For $1 < x < 3$, $\frac{dx}{dt} < 0$
- For $x > 3$, $\frac{dx}{dt} > 0$

Drawing a phase diagram (a number line with left or right arrows indicating the sign of $\frac{dx}{dt}$ in different regions) shows us that $x = -1$ is unstable, $x = 1$ is stable, and $x = 3$ is unstable.