First, we’ll define a regular transition matrix. Remember that the entries in each column must be all positive and sum to one.

Let’s look at a few powers of the matrix A.

Just as we predicted from our analysis in class, as t becomes very large, the columns of $A^t$ are starting to look exactly the same!

Let’s see if the column vector that’s appearing really is an eigenvector with eigenvalue one. We can extract the first column of $A^{100}$ by deleting the second and third columns.

We get $v$ back, verifying that as $t$ increases, $A^t$ approaches a matrix with identical column vectors, where the column vector is an eigenvector with corresponding eigenvalue of one.

We can find the limiting matrix directly by finding the eigenvector that corresponds to the eigenvalue of one. This saves us from computing all of the matrix products. (Of course with Maple, computing these products isn’t very hard at all, but we never really know how big a value of $t$ we will need to get a good answer. So, if we want an exact answer, we should use the eigenvector approach.)

You pick things out of the object above systematically, using indices to work through the nesting of brackets:

We see that the eigenvector corresponding to the eigenvalue of one is not the same vector that we found in our matrix. Why not? If we add up the entries in the eigenvector calculated by Maple we get:

which is not equal to one. So, this eigenvector has not been normalized with respect to the sum. If we were to normalize it, we would get the exact column vector that appeared as we took higher and higher powers of the matrix $A$. I won’t rescale the vector here, as you’ll need to figure out how to do that as part of your homework.
We could have also found the eigenvector by solving the system $Av = \lambda v$, which is equivalent to $(Av - \lambda v) = 0$, or $(A-I)v = 0$. We have developed several methods to solve this kind of system. First, we could row reduce the matrix $(A-I)$ and read off the general form for vectors in the kernel.

```plaintext
> Identity := matrix(3,3,[[1,0,0],[0,1,0],[0,0,1]]);
Anew := evalm(A-Identity);
rref(Anew);
```

We can copy and paste these values into a vector and then verify that $Av = \lambda v$.

```plaintext
> vrref := matrix(3,1,[[3.714285714,1.142857143,1]]);
evalm(A*vrref);
```

Alternatively, we could use the `nullspace` command to find the kernel of the matrix $(A-I)$.

```plaintext
> vkernel := nullspace(Anew);
> evalm(A*vkernel[1]);
```

Now, let’s look at some specific initial conditions and see what happens over time.

```plaintext
> x0_1 := matrix(3,1,[[500],[200],[300]]);
> x0_2 := matrix(3,1,[[0],[800],[200]]);
> x1_1 := evalm(A*x0_1);
x1_2 := evalm(A*x0_2);
> x2_1 := evalm(A*x1_1);
x2_2 := evalm(A*x1_2);
> x10_1 := evalm(A10*x0_1);
x10_2 := evalm(A10*x0_2);
> x100_1 := evalm(A100*x0_1);
x100_2 := evalm(A100*x0_2);
```

Hopefully, you’ve noticed that the two vectors are starting to look very similar to one another. Let’s go ahead and look at time 100.

```plaintext
> x100_1 := evalm(A100*x0_1);
x100_2 := evalm(A100*x0_2);
```

Here, we see that the states at time 100 are almost exactly the same. What will happen after a long time? We can use the sum normalized eigenvector to calculate the exact numbers we expect to see after a long time has passed. Remember that we have not actually normalized our eigenvector so I won’t complete the calculation here. In practice (and on your homework), you would use the fractions given in the eigenvector to divide up the total population size of 1,000 (or whatever the given total size is). Your answer should be pretty close to (if not exactly the same as) what we found using $A^{100}$.

What if our initial population size had been 100 instead of 1,000?

```plaintext
> x0_3 := matrix(3,1,[[100],[0],[0]]);
x1_3 := evalm(A*x0_3);
x2_3 := evalm(A*x1_3);
x10_3 := evalm(A10*x0_3);
x100_3 := evalm(A100*x0_3);
```

Here, we see the same numbers showing up in our final vector, but everything is scaled down by a power of 10 since our total population is ten times smaller.
1.) Most long-distance telephone service in the United States is provided by three companies: AT&T, MCI and Sprint. The three companies are in fierce competition, offering discounts or even cash to those who switch. If the figures advertised by the companies are to be believed, people are switching their long distance provider from one month to the next according to the following percentages:
   - 20% switch from MCI to Sprint
   - 10% switch from Sprint to MCI
   - 50% switch from Sprint to AT&T
   - 20% switch from AT&T to Sprint
   - 10% switch from MCI to AT&T
   - 20% switch from AT&T to MCI
Define the state vector:

\[ x(t) := \begin{bmatrix} a(t) \\ m(t) \\ s(t) \end{bmatrix} \]

where \( a(t) \) is the fraction using AT&T, \( m(t) \) is the fraction using MCI and \( s(t) \) is the fraction using Sprint.

a.) Find the matrix \( A \) such that \( x(t+1) = Ax(t) \), assuming that the customer base remains unchanged. Verify that \( A \) is a regular transition matrix by checking that the criteria in the definition hold.

b.) Find the long term fraction of customers that will be using the services of each company. Your answer should be for the general case where no initial conditions are specified and should be the exact answer in the limit as time goes to infinity. (Do not compute the powers of a matrix for this part.)

c.) For the following three initial conditions, find the number of customers using each of the three services after one month, one year and ten years.

\[ x_0_1 := \begin{bmatrix} 200 \\ 200 \\ 600 \end{bmatrix} \]

\[ x_0_2 := \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} \]

\[ x_0_3 := \begin{bmatrix} 100 \\ 100 \end{bmatrix} \]
d.) For the same three initial conditions as in part (c), find the number of customers using each of the three services after a very, very long time. You should use your result from part (b) here.

2.)

a.) Form a 6 x 6 matrix A by writing the integers 1, 2, 3, 4, 5, and 6 into each column of the matrix in any order you like. Each column must contain all six of the integers. The order of the six integers should be different for at least two of the columns, and preferably will be different for all of the columns. It's fine for a given integer to appear twice in one row. Here is an example of a 4 x 4 matrix that fits the same criteria for the integers 1, 2, 3 and 4.

\[
A := \begin{bmatrix}
2 & 4 & 3 & 1 \\
3 & 1 & 2 & 4 \\
1 & 2 & 1 & 3 \\
4 & 3 & 4 & 2
\end{bmatrix}
\]

b.) Calculate the following powers of your matrix: A^2, A^5, A^10, A^15.
(When you calculate the larger powers, make sure your window is large enough so that you can see the entire matrix at one time.)

c.) Rewrite your matrix A as a product of a constant, k, and a regular transition matrix B. Your answer should have the form A = kB, where B is a regular transition matrix.

d.) Did you see any sort of pattern arising in your calculations in part (b)? What seems to be happening to the columns of each matrix as the power increases? Use the formula you wrote down in part (c) and what you learned in class about transition matrices to explain what you see happening. Write down a formula for A^t as t becomes very large.