Section 4.1

(2, 8) $f'(t) = -3e^{-3t} < 0$ for all values of $t$. Therefore, the function is always decreasing. There are no local maxima or local minima.

(4, 10) $f'(t) = 1 > 0$ for all values of $t$. Therefore, the function is always increasing. There are no local maxima or local minima.

(14) First, we find the local maxima and minima of $f(x) = 3x^2 - 3x + 1$. $f'(x) = 6x - 3 = 3(2x - 1)$. The derivative equals zero for $x = 1/2$. The derivative is always defined and we have no endpoints since we are looking on the domain of all real numbers, $-\infty < x < \infty$. Therefore, the only extreme point is at $x = 1/2$. We see that for $x < 1/2$, $f'(x) < 0$ and for $x > 1/2$, $f'(x) > 0$. Therefore, the function is decreasing for all $x < 1/2$ and increasing for all $x > 1/2$, so $x = 1/2$ is both a local and global minimum point. The value of this global minimum is $f(1/2) = 3(1/4) - 3(1/2) + 1 = 3/4 - 3/2 + 1 = 1/4$. The global minimum value of $f(x)$ is $1/4$, which is greater than zero. Therefore, the function is never negative.

(19) We are told that the sides and the bottom of the can cost $0.12$ cents per square inch, and the top costs $0.70 + 0.12 = 0.82$ cents per square inch. The area of the top and bottom is $\pi r^2$, and the area of the sides is $2\pi rh$. The total volume is $63\text{in}^3$, so $V = \pi r^2 h = 63 \Rightarrow h = 63/(\pi r^2)$. The domain of the cost function will be $1.4 \leq r \leq 1.7$. The total cost is given by the equation

$$C(r) = (.82)\pi r^2 + (.12)\pi r^2 + (.12)2\pi r \left( \frac{63}{\pi r^2} \right)$$

$$C(r) = (.94)\pi r^2 + \frac{15.12}{r}$$

$$C'(r) = 1.88\pi r - \frac{15.12}{r^2} = \frac{1}{r^2} (1.88\pi r^3 - 15.12)$$

The derivative does not exist at $r = 0$, which is not in the given domain. The derivative equals zero when

$$r^3 = \frac{15.12}{1.88\pi}$$

$$r \approx 1.37$$
This point is not in the given domain either. Let \( r^* \) represent the exact value of \( r \) found above. Then, we notice that for \( 0 < r < r^* \), \( C'(r) < 0 \), and for \( r^* < r \), \( C'(r) > 0 \). Therefore, for the general function \( C(r) \) on the domain of all positive real numbers, the critical point \( r = r^* \) is a global minimum point. Hence, on the given domain \( 1.4 \leq r \leq 1.7 \), the function \( C(r) \) is always increasing, so the minimum must occur at the left endpoint, which is \( r = 1.4 \). The height that corresponds to this radius is \( h = \frac{63}{(1.4^2 \pi)} \approx 10.23 \). At this point the total cost is \( C(1.4) \approx 17 \) cents per can.

**Firebreaks**

(ai) The area of one firebreak is \((.01)(50) = 0.5\). Let \( n \) represent the number of firebreaks. We must have \( n \geq 0 \), \( n \) is an integer and \( n \leq 50/.01 = 5000 \). The total area of all firebreaks is \(.5n\). The area of one stand of trees is \((50/n - .01)(50)\). Therefore, the total area lost in the case of a fire is

\[
A(n) = \frac{2500}{n} - .5 + .5n
\]

\[
A'(n) = -\frac{2500}{n^2} + .5
\]

The derivative does not exist when \( n = 0 \), which is also an endpoint. The derivative is zero when

\[
n^2 = \frac{2500}{.5} = 5000
\]

\[
n = \sqrt{5000} \approx 70.7
\]

We see that when \( 0 < n < \sqrt{5000} \), \( A'(n) < 0 \), and when \( \sqrt{5000} < n < 5000 \), \( A'(n) > 0 \). Therefore, \( n = \sqrt{5000} \) is both a local and global minimum on the given domain. Firefighters should therefore construct 71 firebreaks to minimize the area lost. This results in a loss of \( A(71) \approx 70.211km^2 \). We can compare this to other values of \( n \) to see that is really is a minimum. For example, \( A(70) \approx 70.214km^2 \) and \( A(72) \approx 70.222km^2 \).

(aii) Let \( b \) be the width of one firebreak. Then, the area of one firebreak is \( 50b \), so the total area of all firebreaks is \( 50bn \). The domain is now integers satisfying \( 0 \leq n \leq 50/b \). The area of one stand of trees is \((50/n - b)(50)\). The total area lost in the case of a fire is

\[
A(n) = \frac{2500}{n} - 50b + 50bn
\]

\[
A'(n) = -\frac{2500}{n^2} + 50b
\]
The derivative is not defined for \( n = 0 \), which is an endpoint. The derivative is equal to zero when

\[
\frac{n^2}{50b} = \frac{50}{b}
\]

\[
n = \sqrt{\frac{50}{b}} = 5\sqrt{\frac{2}{b}}
\]

Again we see that for \( 0 < n < \sqrt{(50/b)} \), \( A'(n) < 0 \), and for \( \sqrt{(50/b)} < n < 50/b \), \( A'(n) > 0 \). Therefore, the point \( n = \sqrt{(50/b)} \) is both a local and global minimum on the given domain. If the width of a firebreak is quadrupled \( (b \rightarrow 4b) \) we see that the optimal number of firebreaks changes from \( \sqrt{(50/b)} \) to \( \sqrt{(50/4b)} = \frac{1}{2}\sqrt{(50/b)} \), so the optimal number of firebreaks is cut in half.

(b) In this case, we must be careful when calculating the total area of all firebreaks, as the horizontal and vertical lines overlap. Let \( n \) be the number of horizontal firebreaks. Due to symmetry, \( n \) will also be the number of vertical firebreaks. The domain of \( n \) will be \( 0 \leq n \leq 50/0.01 = 5000 \). The area of one firebreak, neglecting the overlap portions, is \( (50 - .01n)(.01) \). The area of one overlap section is \((.01)(.01)\). Therefore, the total area of the firebreaks is \( 2n(50 - .01n)(.01) + n^2(.01)(.01) \). The area of one stand of trees is \( (50/n - .01)(50/n - .01) \). The total area lost is

\[
A(n) = \left(\frac{50}{n} - .01\right)^2 + 2n(50 - .01n)(.01) + (.01)^2n^2
\]

\[
A(n) = \frac{2500}{n^2} - \frac{1}{n} + .0001 + n - .0002n^2 + .0001n^2
\]

\[
A'(n) = \frac{-5000}{n^3} + \frac{1}{n^2} + 1 - .0002n
\]

\[
A'(n) = \frac{1}{n^3}(-5000 + n + n^3 - .0002n^4)
\]

As before, the derivative does not exist when \( n = 0 \), which is an endpoint. Plotting the numerator of the derivative function on your calculator, you should find that it is zero when \( n \approx 17.1 \) and \( n = 5000 \). This second point is also an endpoint. Once you know that \( n = 5000 \) is one zero of the numerator, you can actually solve for the other zero exactly by factoring.

\[-.0002n^4 + n^3 + n - 5000\]
\[= \frac{-1}{5000} \left(n^4 - 5000n^3 - 5000n + (5000)^2\right)\]
\[= \frac{-1}{5000} (n - 5000)(n^3 - 5000)\]

Therefore, the other root satisfies \(n^3 = 5000\), or \(n \approx 17.0998\) as we predicted by looking at the graph. From both the graph and the factored equation above, we see that for \(0 < n < (5000)^{1/3}\) the derivative is negative and for \((5000)^{1/3} < n < 5000\) the derivative is positive. Therefore, the point \(n \approx 17.1\) is both a local and global minimum. Thus, the firefighters should create 17 firebreaks. If there is a fire, the total area lost will be \(A(17) \approx 25.563\). To see that this is really a minimum we compare to two nearby values: \(A(16) \approx 25.678\) and \(A(18) \approx 25.628\).