Section 3.2) 8, 14, 18, 28, 33, 40, 48

(8) We must solve the following system for \( c_1, c_2 \) and \( c_3 \).

\[
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
3 & 4
\end{bmatrix} c_1 + \begin{bmatrix} 2 \\
3 \\
4
\end{bmatrix} c_2 + \begin{bmatrix} 3 \\
4
\end{bmatrix} c_3 = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 3 \\
0 & -1 & -2
\end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\
0 & 1 & 2
\end{bmatrix}
\]

If \( c_1 = c_3 \) and \( c_2 = -2c_3 \), the equation will be satisfied. Letting \( c_3 = 1 \), we have

\[
\begin{bmatrix}
1 & 2 \\
2 & 3 \\
3 & 4
\end{bmatrix} - 2 \begin{bmatrix} 2 \\
3 \\
4
\end{bmatrix} + 1 \begin{bmatrix} 3 \\
4
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

(14)

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 2 \\
0 & 0 & 3
\end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The rank of the matrix is 3, so the kernel of the matrix consists of only the zero vector. Therefore, the columns of the matrix are independent.

(18)

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 4 \\
1 & 3 & 7 \\
1 & 4 & 10
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 2 & 6 \\
0 & 3 & 9
\end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The kernel of the matrix contains some non-zero vectors. Therefore, the columns of the matrix are dependent.

(28)

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 5 \\
1 & 3 & 7
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 4 \\
0 & 2 & 6
\end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\
0 & 1 & 4 \\
0 & 0 & -2
\end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The column vectors of the matrix are linearly independent. Therefore, one possible basis for the image of the matrix is

\[
\left\{ \begin{bmatrix} 1 \\
1 \\
1 \
\end{bmatrix}, \begin{bmatrix} 1 \\
2 \\
3 \
\end{bmatrix}, \begin{bmatrix} 1 \\
5 \\
7 \
\end{bmatrix} \right\}
\]

Notice that the image of the matrix is all of \( \mathbb{R}^3 \), so another possible basis would be \( \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \} \).
(33) The given matrix is already in reduced row echelon form. By inspection, we see that the third column is two times the second, and the fifth column is three times the second plus four times the fourth. The second, fourth, and sixth columns are linearly independent as they are the standard vectors $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ in $\mathbb{R}^4$. Therefore, one possible basis for the image is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$ 

We do not include the zero vector in a basis as it can be written as a scalar multiple of any other vector in the basis (take the scalar equal to zero). Also, notice that $\vec{0}$ and $\vec{v}$ are not independent as $k \vec{0} + 0 \vec{v} = \vec{0}$ for any scalar $k$.

(40) Let $A$ be $m \times n$ and $B$ be $n \times p$, where the columns of both are linearly independent. We must show that the only solution to the linear system $AB\vec{x} = \vec{0}$ is the zero vector. Using the fact that matrix multiplication is associative, we can write $A(B\vec{x}) = \vec{0}$. Then, the vector $B\vec{x}$ must be in the kernel of the matrix $A$. But, since the columns of $A$ are linearly independent, the only vector in the kernel of $A$ is the zero vector, so $B\vec{x} = \vec{0}$. Because the columns of $B$ are linearly independent, the only solution to this last linear system is the zero vector, so $\vec{x} = \vec{0}$. Therefore, the only solution to the original system $AB\vec{x} = \vec{0}$ is the zero vector. Thus, $\ker(AB) = \{\vec{0}\}$ and the columns of the matrix product $AB$ are linearly independent.

(48) The plane $E$ in $\mathbb{R}^3$ defined by $3x_1 + 4x_2 + 5x_3 = 0$ is equivalent to the kernel of the matrix $A = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$. To find the matrix $B$ whose image is $E$, we treat the equation of the plane as if it is the relation resulting from the rref($B$), or $x_1 = -\frac{4}{3}x_2 - \frac{5}{3}x_3$, where $x_2$ and $x_3$ are free variables. This suggests two vectors for the basis of the image of $B$, $\begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$.

Then, we define $B$ to be the matrix with these vectors as its columns, or $B = \begin{bmatrix} -4 & -5 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}$. 

2
Section 3.3) 6, 7, 21, 30, 32, 36, 37

(6) We must solve \( A\vec{x} = \vec{0}. \)

\[
\begin{bmatrix}
1 & 3 & 2 \\
1 & 2 & 3 \\
1 & 1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 2 \\
0 & -1 & 1 \\
0 & -2 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & -1 \\
0 & 0 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The only vector in the kernel is the zero vector, so the basis for the kernel is the empty set \( \emptyset \) and \( \dim(\ker(A))=0. \)

(7) We must solve \( A\vec{x} = \vec{0}. \)

\[
\begin{bmatrix}
0 & 1 & 2 & 0 & 3 \\
0 & 0 & 1 & 4
\end{bmatrix}
\]

The matrix is already in reduced row echelon form, so we can read off the form of the solution vectors: \( x_2 = -2x_3 - 3x_5 \) and \( x_4 = -4x_5. \) There are three free variables: \( x_1, x_3 \) and \( x_5. \) Solutions are of the form

\[
\vec{x} = r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]

The kernel has dimension 3 and a basis is

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix}
\]

(21)

\[
\begin{bmatrix}
1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 0 & -2 & 2 \\
1 & -1 & -2 & 0 & 3 \\
2 & -2 & -1 & 3 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & -1 & 1 & 1 \\
0 & 0 & -1 & -1 & 3 \\
0 & 0 & -1 & -1 & 2 \\
0 & 0 & 1 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 0 & 2 & -2 \\
0 & 0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -1 & 0 & 2 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
All vectors in the kernel are of the form

\[ \vec{x} = r \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \]

where \( r \) and \( s \) are scalars. Therefore, \( \dim(\ker(A))=2 \) and a basis for the kernel is

\[ \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \]

Columns 1, 3 and 5 are pivot columns of \( A \), so \( \dim(\text{im}(A))=3 \) and a basis for the image is

\[ \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\} \]

Notice that \( \dim(\text{im}(A))+\dim(\ker(A))=3+2=5 \) which is the number of columns in \( A \).

(30) All vectors in the subspace of \( \mathbb{R}^4 \) defined by \( 2x_1 - x_2 + 2x_3 + 4x_4 = 0 \) must satisfy \( x_1 = \frac{1}{2}x_2 - x_3 - 2x_4 \), or have the form

\[ \vec{x} = r \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \]

Therefore, one possible basis is

\[ \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

Looking at the second, third and fourth entries, we see that only one of the three vectors has a non-zero entry in these positions, guaranteeing the linear independence of the vectors.
(32) All elements, $\vec{x}$, of the subspace must satisfy $[1\ 0\ -1\ 1] \cdot \vec{x} = 0$ and $[0\ 1\ 2\ 3] \cdot \vec{x} = 0$. The vectors, $\vec{x}$, that we are looking for are in the kernel of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

or are of the form

$$\vec{x} = r \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, one possible basis is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(36) No, you cannot find a 3 x 3 matrix $A$ such that the $\text{im}(A) = \ker(A)$. The Rank-Nullity Theorem says that $\text{nullity}(A) + \text{rank}(A)$ is equal to the number of columns of $A$, or in this case 3. If the image of $A$ and the kernel of $A$ are the same, they must have the same number of vectors in their basis. In other words, the nullity and the rank must be equal, implying that they are both equal to 3/2, which is not possible. In general, for this to be true, the matrix must have an even number of columns.

(37) For the given matrix, $\text{im}(A)$ has dimension 2 with basis $\{\vec{e}_1, \vec{e}_2\}$. Therefore, the kernel of $A$ must have dimension $5 - 2 = 3$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$