Section 8.2) 2, 4, 6, 8, 11, 15, 16

(2) \[ A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \]

(4) \[ A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \]

The eigenvalues of \( A \) are the zeros of \( f_A(\lambda) = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2) \), or \( \lambda_1 = 2 \) and \( \lambda_2 = 7 \). Both eigenvalues are positive, so the matrix \( A \) is positive definite.

(6) \[ A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \]

Here we notice that \( \det(A) = -1 \). Since \( A \) is a 2 x 2 matrix, there are two eigenvalues. We also know that \( \det(A) = \lambda_1 \cdot \lambda_2 \). Since the product of the eigenvalues is negative, we know that one must be negative and one must be positive. Therefore, the matrix \( A \) is indefinite.

(8) Let \( \vec{v} \) be an eigenvector of the matrix \( A \) with associated eigenvalue \( \lambda \). Then, \( A^2 \vec{v} = A(\lambda \vec{v}) = \lambda^2 \vec{v} \). Therefore, the eigenvalues of the matrix \( A^2 \) are the squares of the eigenvalues of the matrix \( A \), and are all positive or zero. Therefore, the matrix \( A^2 \) is either positive definite or positive semidefinite. The matrix \( A^2 \) is positive definite if and only if zero is not an eigenvalue of \( A \), or \( A \) is invertible.

(11) If \( A \) is invertible and \( \lambda \) is an eigenvalue of \( A \) with associated eigenvector \( \vec{v} \), then \( A^{-1} \vec{v} = (1/\lambda) \vec{v} \). In other words, the eigenvalues of \( A^{-1} \) are the reciprocals of the eigenvalues of \( A \). Therefore, their eigenvalues have the same sign, so the matrices \( A \) and \( A^{-1} \) must have the same definiteness.

(15) From problem (4) we have \[ A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \]

with eigenvalues \( \lambda_1 = 2 \) and \( \lambda_2 = 7 \). By Fact 8.2.2 \( q(\vec{x}) = 2c_1^2 + 7c_2^2 \) = 1 where \( c_1 \) and \( c_2 \) are the coordinates of \( \vec{x} \) with respect to the orthonormal eigenbasis for \( A \). This equation
describes an ellipse on the $c_1$-$c_2$ plane. Let’s find the eigenvectors that define this new coordinate system.

\[ E_2 = \ker \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} = \sp \begin{bmatrix} 1 \\ -2 \end{bmatrix} \]

\[ E_7 = \ker \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = \sp \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

The curve will be an ellipse centered over the lines $y = -2x$ and $y = x/2$ and flattened along the line $y = -2x$.

(16) From problem (2) we have

\[ A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \]

The eigenvalues are the zeros of the equation $f_A(\lambda) = \lambda^2 - 1/4 = (\lambda - 1/2)(\lambda + 1/2)$. We have $\lambda_1 = 1/2$ and $\lambda_2 = -1/2$. Fact 8.2.2 tells us that $q(\vec{x}) = (c_1^2 - c_2^2)/2 = 1$ where $c_1$ and $c_2$ are the coordinates of $\vec{x}$ with respect to the orthonormal eigenbasis for $A$. The equation $c_1^2 - c_2^2 = 2$ describes a hyperbola with intercepts on the $c_1$ axis. Let’s find the eigenvectors that define this new coordinate system.

\[ E_{1/2} = \ker \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \sp \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ E_{-1/2} = \ker \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} = \sp \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

The curve will be a hyperbola centered over the lines $y = x$ and $y = -x$ with intercepts on the line $y = x$. 