Section 7.5 2, 3, 4, 5, 9, 24, 45, 46

(2) We factor \( z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z - i)(z + i) \). This gives us four solutions: \( \pm 1 \) and \( \pm i \). When we graph these on the complex plane, we notice that we have four vectors of length one separated by 90°.

(3) We first write \( z \) in polar coordinates as \( z = r(\cos \phi + i \sin \phi) \). Then, we solve \( z^n = r^n(\cos (\phi n) + i \sin (\phi n)) = 1 \). This is true if \( r^n = 1 \), \( \cos (\phi n) = 1 \) and \( \sin (\phi n) = 0 \). It follows that \( r = 1 \) and \( (\phi n) \) is a multiple of \( 2\pi \), or \( \phi = \frac{2k\pi}{n} \) where \( k \) is an integer between 0 and \( n - 1 \). Then, \( z = \cos \left( \frac{2k\pi}{n} \right) + i \sin \left( \frac{2k\pi}{n} \right) \). Graphically, this gives us points evenly spaced around the unit circle. The angle between neighboring points is \( \frac{2\pi}{n} \).

(4,5) Let \( z = r(\cos \phi + i \sin \phi) \). Then, the two solutions to \( w^2 = z \) are \( \sqrt{r}(\cos \left( \frac{\phi}{2} \right) + i \sin \left( \frac{\phi}{2} \right)) \) and \( \sqrt{r}(\cos \left( \frac{\pi + \phi}{2} \right) + i \sin \left( \frac{\pi + \phi}{2} \right)) \). Solutions to \( w^n = z \) are \( r^{1/n}(\cos \left( \frac{2k\pi + \phi}{n} \right) + i \sin \left( \frac{2k\pi + \phi}{n} \right)) \) where \( k \) is all integers between 0 and \( n - 1 \).

(9) \( z = 0.8 - 0.7i = r(\cos \phi + i \sin \phi) \) where \( r = \sqrt{0.8^2 + 0.7^2} = \sqrt{1.13} \) and \( \phi = \tan^{-1}(-0.875) \approx -41^\circ \). The powers \( z^2, z^3, \ldots \) will be progressively longer and will rotate approximately \( 41^\circ \) clockwise each step. The end result will be an outward spiral.

(24) To find the eigenvalues we solve \( \det(\lambda I - A) = 0 \).

\[
\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -5 & 7 & \lambda - 3 \end{bmatrix} = \lambda [\lambda(\lambda - 3) + 7] + 1 [0 - 5] = \lambda^3 - 3\lambda^2 + 7\lambda - 5 = (\lambda - 1)(\lambda^2 - 2\lambda + 5)
\]

From this factorization we see that one eigenvalue is \( \lambda_1 = 1 \). To find the other two eigenvalues we use the quadratic formula:

\[
\lambda_{2,3} = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i
\]

(45) We must find the eigenvalues and their geometric multiplicities. The characteristic polynomial is \( f_A(\lambda) = \lambda^2 - 2\lambda + (1 - a) \). We use the quadratic formula:

\[
\lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4 + 4a}}{2} = 1 \pm \sqrt{a}
\]

We have two cases to look at. If \( a \neq 0 \), then there are two distinct roots (possibly complex) and the matrix is diagonalizable. For the case \( a = 0 \), the eigenvalue \( \lambda = 1 \) has algebraic
multiplicity of 2. We must look at the associated eigenspace and determine its dimension.

\[ E_1 = \ker \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \sp \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

The dimension of \( E_1 \) is one, so there is no eigenbasis for \( A \) in this case. Therefore, the matrix is diagonalizable only if \( a \neq 0 \).

(46) The characteristic polynomial is \( f_A(\lambda) = \lambda^2 + a^2 \). If \( a \neq 0 \), there are two distinct eigenvalues, \( \lambda_{1,2} = \pm ia \), and the matrix is diagonalizable. If \( a = 0 \), there is one eigenvalue, \( \lambda = 0 \) with algebraic multiplicity of 2. We look at the eigenspace \( E_0 \).

\[ E_0 = \ker \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \sp \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \]

The dimension of \( E_0 \) is two, so the matrix is diagonalizable. (Notice that the zero matrix is already in the desired diagonal form.) Therefore, the matrix is always diagonalizable.

**Section 7.6** 2, 4, 22, 34

(2) The matrix is diagonal so the eigenvalues are the diagonal entries \( \lambda_1 = -1.1 \) and \( \lambda_2 = 0.9 \). One of these eigenvalues has modulus greater than one, so, by Fact 7.6.2, the zero state is not stable.

(4) From example 2, we know that the eigenvalues are \( \lambda_{1,2} = 0.9 \pm i0.4 \). The modulus of these eigenvalues is \( \sqrt{0.9^2 + 0.4^2} = \sqrt{0.97} < 1 \). Therefore, by Fact 7.6.2, the zero state is stable.

(22) The characteristic polynomial of \( A \) is \( f_A(\lambda) = \lambda^2 + 4\lambda + 13 \). Using the quadratic formula we have:

\[ \lambda_{1,2} = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i \]

In polar form:

\[ -2 \pm 3i = \sqrt{13}(\cos \phi \pm i \sin \phi) \]

where \( \phi = \tan^{-1}(-3/2) \approx 124^\circ \). (Notice that we want the angle to be in the second quadrant since (for \( \lambda_1 \) the real part is negative and the imaginary part is positive.) We must find the eigenspace for \( E_{-2+3i} \).

\[ E_{-2+3i} = \ker \begin{bmatrix} -9 + 3i & 15 \\ -6 & 9 + 3i \end{bmatrix} = \sp \begin{bmatrix} 15 \\ 9 - 3i \end{bmatrix} = \sp \begin{bmatrix} 5 \\ 3 - i \end{bmatrix} \]
We write this vector as a sum of its real and imaginary parts:

\[
\begin{bmatrix}
5 \\
3 - i
\end{bmatrix} = \begin{bmatrix}
5 \\
3
\end{bmatrix} + i \begin{bmatrix}
0 \\
-1
\end{bmatrix}
\]

Then, we use Fact 7.6.3.

\[
\vec{x}(t) = \left(\sqrt{13}\right)^t \begin{bmatrix}
0 & 5 \\
-1 & 3
\end{bmatrix} \begin{bmatrix}
\cos(\phi t) & -\sin(\phi t) \\
\sin(\phi t) & \cos(\phi t)
\end{bmatrix} \begin{bmatrix}
3/5 & -1 \\
1/5 & 0
\end{bmatrix} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

\[
= \left(\sqrt{13}\right)^t \begin{bmatrix}
5 \sin(\phi t) & 5 \cos(\phi t) \\
-\cos(\phi t) + 3 \sin(\phi t) & \sin(\phi t) + 3 \cos(\phi t)
\end{bmatrix} \begin{bmatrix}
-1 \\
0
\end{bmatrix} = \left(\sqrt{13}\right)^t \begin{bmatrix}
-5 \sin(\phi t) \\
\cos(\phi t) - 3 \sin(\phi t)
\end{bmatrix}
\]

where \(\phi = \tan^{-1}(-3/2) \approx 124^\circ\). If we plot the trajectory, we will see an ellipse spiraling outward.

(34) If \(|\det(A)| \geq 1\), then \(|\lambda_1 \lambda_2 \cdots \lambda_n| \geq 1\) (by Fact 7.5.5). Therefore, the modulus of at least one of the eigenvalues is greater than or equal to one. Hence, the zero state is not stable. If \(|\det(A)| < 1\), then \(|\lambda_1 \lambda_2 \cdots \lambda_n| < 1\) (by Fact 7.5.5). In this case, we cannot determine if the zero state is stable. It is possible that the modulus of all of the eigenvalues is less than one, but it is also possible that one has length greater than one (say 2) and the other has length less than one (say 0.1) giving a product less than one (in our example, 0.2).