Section 7.4) 10, 12, 22, 32, 34, 37, 38, 48, 50

(10) \[ \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 3 & -4 \\ 1 & \lambda + 1 \end{bmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \]

The eigenvalue \( \lambda = 1 \) has algebraic multiplicity of two. Now, we find the eigenvectors by finding \( \ker(I - A) \).

\[ \ker \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \]

The eigenspace, \( E_1 \) has basis, \( \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \) of dimension one. Therefore, there is no eigenbasis and the matrix \( A \) is not diagonalizable.

(12) The given matrix is upper triangular, so the eigenvalues are the diagonal entries. \( \lambda_1 = 2 \) has algebraic multiplicity of one and \( \lambda_2 = 1 \) has algebraic multiplicity of two. Now, we find the eigenvectors by finding \( \ker(2I - A) \) and \( \ker(I - A) \). First, for \( \lambda_1 = 2 \):

\[ \ker \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

The eigenspace, \( E_2 \) has basis, \( \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \) of dimension one. Now, for \( \lambda_2 = 1 \):

\[ \ker \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

The eigenspace, \( E_1 \) has basis, \( \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \) of dimension two. Therefore, \( A \) is diagonalizable with:

\[ S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
(22) Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = b$. If $b \neq 1$, there are two distinct eigenvalues, and two distinct eigenvectors and the matrix is diagonalizable. If $b = 1$, we must look at the associated eigenspace, $E_1$ and determine when it has dimension two.

$$E_1 = \ker \begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix}$$

If $a = 0$, $E_1 = \mathbb{R}^2$ and the matrix is diagonalizable. If $a \neq 0$, $E_1 = \text{sp} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the matrix is not diagonalizable.

(32) $A$ is $2 \times 2$, so the characteristic polynomial is $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \text{det}(A) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. So, $\lambda_1 = 2$ and $\lambda_2 = 3$. We find bases for the eigenspaces $E_2$ and $E_3$.

$$E_2 = \ker \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} = \text{sp} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad E_3 = \ker \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} = \text{sp} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore,

$$A^t = SD^tS^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^t & 0 \\ 0 & 3^t \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2^t + 2(3^t) & 2(2^t) - 2(3^t) \\ -2^t + 3^t & 2(2^t) - 3^t \end{bmatrix}$$

Then,

$$A^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2^t + 2(3^t) & 2(2^t) - 2(3^t) \\ -2^t + 3^t & 2(2^t) - 3^t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2^t + 2(3^t) + 4(2^t) - 4(3^t) \\ -2^t + 3^t + 4(2^t) - 2(3^t) \end{bmatrix} = \begin{bmatrix} 2(2^t - 3^t) \\ 3(2^t) - 3^t \end{bmatrix}$$

(34) $A$ is $2 \times 2$, so the characteristic polynomial is $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \text{det}(A) = \lambda^2 - 5\lambda/4 + 1/4 = (\lambda - 1/4)(\lambda - 1)$. So, $\lambda_1 = 1/4$ and $\lambda_2 = 1$. We find bases for the eigenspaces $E_{1/4}$ and $E_1$.

$$E_{1/4} = \ker \begin{bmatrix} -1/4 & -1/4 \\ -1/2 & -1/2 \end{bmatrix} = \text{sp} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad E_1 = \ker \begin{bmatrix} 1/2 & -1/4 \\ -1/2 & 1/4 \end{bmatrix} = \text{sp} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore,

$$A^t = SD^tS^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (1/4)^t & 0 \\ 0 & 1^t \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2(1/4)^t + 1 & -1/4^t + 1 \\ -2(1/4)^t + 2 & (1/4)^t + 2 \end{bmatrix}$$

Then,

$$A^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2(1/4)^t + 1 & -1/4^t + 1 \\ -2(1/4)^t + 2 & (1/4)^t + 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
\(T(50)

We are given \(T P\) equal to three which is the dimension of \(\lambda\). Therefore, \(E \lambda\) leading to three eigenvalues: 

\[
\frac{1}{3} \begin{bmatrix}
2(1/4)^t + 1 - 2(1/4)^t + 2 \\
-2(1/4)^t + 2 + 2(1/4)^t + 4
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
3 \\
6
\end{bmatrix} = \begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

(37) If \(A\) and \(B\) are two \(2 \times 2\) matrices with \(\det(A) = \det(B) = \text{tr}(A) = \text{tr}(B) = 7\), \(A\) and \(B\) must be similar. First, they have the same characteristic polynomial \(f_A(\lambda) = f_B(\lambda) = \lambda^2 - 7\lambda + 7\). This polynomial has two distinct real roots, so there are two distinct eigenvalues. It follows that there are two eigenvectors and thus, \(A\) and \(B\) must be diagonalizable. \(A\) and \(B\) are both similar to their diagonal matrix \(D\) (that contains the eigenvalues on the diagonal), so they must be similar to one another.

(38) If \(A\) and \(B\) are two \(2 \times 2\) matrices with \(\det(A) = \det(B) = \text{tr}(A) = \text{tr}(B) = 4\), then \(A\) and \(B\) are not necessarily similar. They must have the same characteristic polynomial \(f_A(\lambda) = f_B(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2\). Therefore, they both have one eigenvalue, \(\lambda = 2\) with algebraic multiplicity of two. It is possible that for one matrix, the geometric multiplicity of \(\lambda = 2\) is one and for the other, it is two. If this is the case, one matrix will be diagonalizable and the other will not. Consider

\[
A = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix} \quad B = \begin{bmatrix}
2 & 1 \\
0 & 2
\end{bmatrix}
\]

(48) We are given \(T(f(x)) = f(2x)\) from \(P_2\) to \(P_2\). For a general polynomial we have \(T(a + bx + cx^2) = a + b(2x) + c(2x)^2 = a + 2bx + 4cx^2\). We must solve \(T(f(x)) = \lambda f(x)\) or \(a + 2bx + 4cx^2 = a\lambda + b\lambda x + c\lambda x^2\). This gives us three separate equations that must be satisfied:

\[
a = a\lambda \\
2b = b\lambda \\
4c = c\lambda
\]

leading to three eigenvalues: \(\lambda_1 = 1\), \(\lambda_2 = 2\) and \(\lambda_3 = 4\). Next, we must find \(E_1\), \(E_2\) and \(E_4\). If \(\lambda = 1\), we must have \(b = c = 0\) to satisfy the second two equations. Therefore, \(E_1 = \text{sp}(1)\). If \(\lambda = 2\), we must have \(a = c = 0\) to satisfy the first and third equations. Therefore, \(E_2 = \text{sp}(x)\). If \(\lambda = 4\), we must have \(a = b = 0\) to satisfy the first and second equations. Therefore, \(E_4 = \text{sp}(x^2)\). \(T\) is diagonalizable as the sum of the geometric multiplicities is equal to three which is the dimension of \(P_2\). Also notice that the standard basis \(\{1, x, x^2\}\) is an eigenbasis for \(T\).

(50) We are given \(T(f(x)) = f(x - 3)\) from \(P_2\) to \(P_2\). For a general polynomial we have \(T(a + bx + cx^2) = a + b(x - 3) + c(x - 3)^2 = a + bx - 3b + cx^2 - 6cx + 9c\)

\[
= (a - 3b + 9c) + (b - 6c)x + cx^2.
\]

We must solve \(T(f(x)) = \lambda f(x)\) or \((a - 3b + 9c) + (b - 6c)x + cx^2 = a\lambda + b\lambda x + c\lambda x^2\).
This gives us three separate equations that must be satisfied:
\[ a - 3b + 9c = a\lambda \]
\[ b - 6c = b\lambda \]
\[ c = c\lambda \]

For the last equation to be satisfied, we must have either \( c = 0 \) or \( \lambda = 1 \). If \( \lambda = 1 \), equation two becomes \( b - 6c = b \) or \(-6c = 0\), so \( c = 0 \). Then, equation one becomes \( a - 3b = a \) so \(-3b = 0\) or \( b = 0 \). Therefore, \( \lambda = 1 \) is an eigenvalue with eigenspace \( E_1 = \text{sp}(1) \). If \( \lambda \neq 1 \), equation three holds only if \( c = 0 \). Then, for equation two to hold, we must have \( b = 0 \). Then, for equation one to hold, \( a = 0 \). This gives us the zero function which is not an eigenfunction. Thus, the only eigenvalue is \( \lambda = 1 \) with eigenfunction 1. The dimension of the eigenspace is only one, so there is no eigenbasis. Therefore, \( T \) is not diagonalizable.