Section 7.3) 8, 18, 20, 23, 28, 31, 36, 38, 44

(8) The matrix is upper triangular, so the eigenvalues are the diagonal entries (Fact 7.2.2). Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. To find their corresponding eigenvectors and eigenspaces we look at the $\ker(\lambda I - A)$.

$$(\lambda_1 I - A) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The geometric multiplicity of $\lambda_1 = 1$ is one and a basis for the eigenspace $E_1$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

$$(\lambda_2 I - A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The geometric multiplicity of $\lambda_2 = 2$ is one and a basis for the eigenspace $E_2$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$$(\lambda_3 I - A) = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The geometric multiplicity of $\lambda_3 = 3$ is one and a basis for the eigenspace $E_3$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

One possible eigenbasis for $A$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(18) The matrix is upper triangular, so the eigenvalues are the diagonal entries (Fact 7.2.2). Eigenvalues are $\lambda_1 = 0$ with algebraic multiplicity of two, and $\lambda_2 = 1$ with algebraic multiplicity of two. To find their corresponding eigenvectors and eigenspaces we look at the $\ker(\lambda I - A)$.

$$(\lambda_1 I - A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
The geometric multiplicity of $\lambda_1 = 0$ is two and a basis for the eigenspace $E_0$ is $\egin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

$$(\lambda_2 I - A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

The geometric multiplicity of $\lambda_2 = 1$ is one and a basis for the eigenspace $E_1$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

The geometric multiplicities of the eigenvalues add up to three, so there is no eigenbasis for $A$.

(20) We first look at the eigenvalue $\lambda_1 = 1$ with algebraic multiplicity of two.

$$(\lambda_1 I - A) = \begin{bmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If $a = 0$, the geometric multiplicity of $\lambda_1 = 1$ is two and a basis for the eigenspace $E_1$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. If $a \neq 0$, the geometric multiplicity of $\lambda_1 = 1$ is one and a basis for the eigenspace $E_1$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Next, we look at the eigenvalue $\lambda_2 = 2$ with algebraic multiplicity of one.

$$(\lambda_2 I - A) = \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -b - ac \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix}$$

The geometric multiplicity of $\lambda_2 = 2$ is one and a basis for the eigenspace $E_2$ is $\begin{bmatrix} b + ac \\ c \\ 1 \end{bmatrix}$.

There is an eigenbasis for $A$ only when $a = 0$. If $a = 0$, the eigenbasis is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} b \\ c \\ 1 \end{bmatrix}$. 

2
(23) The matrix is upper triangular, so the eigenvalues are the diagonal entries (Fact 7.2.2). The only eigenvalue is \( \lambda = 1 \) with algebraic multiplicity of two. To find the corresponding eigenvectors and eigenspaces we look at the \( \ker(I - A) \).

\[
(I - A) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}
\]

The geometric multiplicity of \( \lambda = 1 \) is one and a basis for the eigenspace \( E_1 \) is \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \). The geometric multiplicity of the eigenvalue is less than two, so there is no eigenbasis for \( A \). The matrix \( A \) represents a shear parallel to the \( x \)-axis. Therefore, it makes sense that the dimension of the eigenspace \( E_1 \) is one as vectors on the \( x \)-axis are transformed to themselves and all other vectors move their tips parallel to the \( x \)-axis (so do not map to multiples of themselves).

(28) The given matrix is upper triangular with all \( \lambda \)'s on the diagonal. Therefore \( \lambda \) is the only eigenvalue with algebraic multiplicity of \( n \). We must find the kernel of \( \lambda I - J_n(\lambda) \). This matrix contains all zeros except for the diagonal directly above the main diagonal. The entries in this diagonal are all negative one. The kernel of this matrix has dimension one, so the geometric multiplicity of \( \lambda \) is one. A basis for the eigenspace \( E_\lambda \) is \( \{ e_1 \} \).

(31) If there is an eigenbasis for a matrix \( A \), then the algebraic and geometric multiplicities of its eigenvalues must be equal (Facts 7.3.3 and 7.3.7).

(36) If matrices \( A \) and \( B \) are similar, \( \det(A) = \det(B) \) and \( \text{tr}(A) = \text{tr}(B) \) (Fact 7.3.8). We check these two facts:

\[
\det \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} = 0 - 5 = -5 \quad \quad \quad \quad \det \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 3 - 8 = -5
\]

\[
\text{tr} \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} = 0 + 3 = 3 \quad \quad \quad \quad \text{tr} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 1 + 3 = 4
\]

The traces of the two matrices are not the same, so the matrices are not similar.

(38) A transformation \( T \) has a fixed point if \( T(\vec{x}) = \vec{x} \), or \( A\vec{x} = \vec{x} \). In other words, the matrix \( A \) has an eigenvalue \( \lambda = 1 \), with corresponding eigenvector equal to the fixed point, \( \vec{x} \). We look at the characteristic polynomial for the matrix \( A \). The linear transformation
is a rotation, so \( \det(A) = 1 \) (Def. 6.3.2). By Fact 7.2.5 the polynomial has the form \( f_A(\lambda) = \lambda^3 - \text{tr}(A)\lambda^2 + \cdots - 1 \). Using this formula we see that \( f_A(0) = -1 \) and \( \lim_{\lambda \to \infty} = \infty \). Therefore, by the Intermediate Value Theorem, the function \( f_A(\lambda) \) crosses the \( \lambda \)-axis, or has a zero on the interval \([0, \infty)\). By Fact 7.1.2, the possible real eigenvalues of the matrix are 1 and -1. The only one of these on the interval \([0, \infty)\) is 1, so the matrix must have the eigenvalue \( \lambda = 1 \). Therefore, there exists a nonzero vector \( \vec{x} \in \mathbb{R}^3 \) such that \( T(\vec{x}) = \vec{x} \).

(44) The diagonal entries of the matrix indicate the amount of pollutant that remains in the same place after one week. For example, \( a_{11} = 0.7 \) means that 70% of the pollutant present in Lake Silvaplana at a given time is still there one week later. The rest is either carried down the river to Lake Sils, is absorbed, or evaporates. The off diagonal entries indicate how much of the pollutant is carried from one lake to the next. For example, \( a_{21} = 0.1 \) means that 10% of the pollutant present in Lake Silvaplana at any given time can be found in Lake Sils one week later. \( a_{32} = 0.2 \) means that 20% of the pollutant present in Lake Sils at any given time can be found in Lake St. Moritz one week later. \( a_{31} = 0 \) means that no pollutant is carried down from Lake Silvaplana to Lake St. Moritz in just one week. The matrix is lower triangular since no pollutant is carried from Lake Sils up to Lake Silvaplana, or from Lake St. Moritz up to Lake Sils as the river flows in the opposite direction.

To find closed formulas for the amount of pollutant in each of the three lakes after \( t \) weeks, we must find the eigenvalues and their corresponding eigenvectors. Since the matrix is lower triangular, the eigenvalues are the diagonal entries, so \( \lambda_1 = 0.7 \), \( \lambda_2 = 0.6 \) and \( \lambda_3 = 0.8 \). We find the kernels of the matrices \((\lambda_i I - A)\).

\[
(\lambda_1 I - A) = \begin{bmatrix} 0 & 0 & 0 \\ -0.1 & 0.1 & 0 \\ 0 & -0.2 & -0.1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}
\]

Eigenvalue \( \lambda_1 = 0.7 \) has eigenvector \( \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \).

\[
(\lambda_2 I - A) = \begin{bmatrix} -0.1 & 0 & 0 \\ -0.1 & 0 & 0 \\ 0 & -0.2 & -0.2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}
\]
Eigenvalue $\lambda_2 = 0.6$ has eigenvector \[
\begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix}.
\]

\[
(\lambda_3 I - A) = \begin{bmatrix}
0.1 & 0 & 0 \\
-0.1 & 0.2 & 0 \\
0 & -0.2 & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Eigenvalue $\lambda_3 = 0.8$ has eigenvector \[
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]

The initial state vector is $\vec{x}_0 = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$. We must find the coordinate vector with respect to the eigenbasis. To do this, one can set up and solve a matrix equation, or simply notice that the first eigenvector is the only one with a nonzero entry in the first row so the coefficient must be 100. Then, use the second eigenvector to cancel the resulting entry in the second row. Lastly, use the third eigenvector to cancel the resulting entry in the third row.

\[
\vec{x}_0 = \begin{bmatrix}
100 \\
0 \\
0
\end{bmatrix} = 100 \begin{bmatrix}
1 \\
1 \\
-2
\end{bmatrix} - 100 \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix} + 100 \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

The closed formulas are:

\[
x_1(t) = 100(0.7)^t(1) - 100(0.6)^t(0) + 100(0.8)^t(0) = 100(0.7)^t
\]

\[
x_2(t) = 100(0.7)^t(1) - 100(0.6)^t(1) + 100(0.8)^t(0) = 100 \left[ (0.7)^t - (0.6)^t \right]
\]

\[
x_3(t) = 100(0.7)^t(-2) - 100(0.6)^t(-1) + 100(0.8)^t(1) = 100 \left[ -2(0.7)^t + (0.6)^t + (0.8)^t \right]
\]

To find the maximum of $x_2(t)$ we use calculus.

\[
\frac{dx_2}{dt} = 100 \left[ (0.7)^t \ln(0.7) - (0.6)^t \ln(0.6) \right] = 0 \Rightarrow \left( \frac{0.7}{0.6} \right)^t = \frac{\ln(0.6)}{\ln(0.7)}
\]
\[ t \ln \frac{0.7}{0.6} = \ln \left( \frac{\ln(0.6)}{\ln(0.7)} \right) \implies t = \frac{\ln \left( \frac{\ln(0.6)}{\ln(0.7)} \right)}{\ln \frac{0.7}{0.6}} \approx 2.33 \]

The pollution in Lake Sils reaches a maximum after approximately 2.33 weeks. Keep in mind however that our model is only accurate for integer values of \( t \), so this result must be used cautiously.