Section 6.1) 15, 16, 20, 33, 34

(15) We must find all of the patterns in the following matrix.

\[
\begin{bmatrix}
0 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
0 & 0 & 0 & 4 \\
0 & 0 & 3 & 4
\end{bmatrix}
\]

The only non-zero pattern is \(\{a_{21}, a_{12}, a_{43}, a_{34}\}\): \(1 \cdot 2 \cdot 3 \cdot 4 = 24\). This has two inversions, so the determinant is 24.

(16) We must find all of the patterns in the following matrix.

\[
\begin{bmatrix}
0 & 0 & 2 & 3 & 1 \\
0 & 0 & 0 & 2 & 2 \\
0 & 9 & 7 & 9 & 3 \\
0 & 0 & 0 & 0 & 5 \\
3 & 4 & 5 & 8 & 5
\end{bmatrix}
\]

The only non-zero pattern is \(\{a_{51}, a_{32}, a_{13}, a_{24}, a_{45}\}\): \(3 \cdot 9 \cdot 2 \cdot 2 \cdot 5 = 540\). This has six inversions, so the determinant is 540.

(20) We notice that all rows of the matrix are the same. Therefore, the product of every pattern is the same and equal to one. We defined the determinant so that if a matrix is not invertible, its determinant is zero. This matrix is obviously not invertible so the determinant must be zero.

(33) If a matrix has a row of zeros or a column of zeros, then all patterns will contain the entry zero, so their products will be zero. Therefore, the determinant must be zero.

(34) There is only one non-zero pattern in a permutation matrix. This pattern is found by taking the entry from each column that contains a one. Therefore, the product of the pattern must be one. Depending on the number of inversions, the determinant will be \(\pm 1\).
Section 6.2) 4, 5, 9, 10, 11, 12, 16, 21, 31, 32, 46

(4) 
\[ \begin{vmatrix} 0 & 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 5 & 3 & 9 & 9 \\ 0 & 7 & 4 & 0 & 1 \\ 3 & 9 & 5 & 4 & 8 \end{vmatrix} = 3 \begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 5 & 3 & 9 & 9 \\ 7 & 4 & 0 & 1 \end{vmatrix} = 3 \begin{bmatrix} 2 & 0 & 1 \\ 5 & 9 & 9 \\ 7 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 9 \\ 7 & 4 & 0 \end{bmatrix} \]

= 6 \begin{bmatrix} 9 & 2 & 1 \\ 7 & 1 \end{bmatrix} - 9 \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = 54 [(2 - 7) - (8 - 7)] = 54(-6) = -324

(5) 
\[ \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 3 & 3 \end{vmatrix} = 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 24 \]

(9) 
\[ \begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 \\ -9\vec{v}_3 \\ \vec{v}_4 \end{vmatrix} = -9 \begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{vmatrix} = -9(8) = -72 \]

(10) 
\[ \begin{vmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{vmatrix} = - \begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{vmatrix} = -8 \]

(11) 
\[ \begin{vmatrix} \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \\ \vec{v}_4 \end{vmatrix} = - \begin{vmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{vmatrix} = 8 \]
(12)
\[
\begin{vmatrix}
\vec{v}_1 & \vec{v}_2 + 9\vec{v}_4 \\
\vec{v}_2 & \vec{v}_3 \\
\vec{v}_3 & \vec{v}_4 \\
\vec{v}_4 & \\
\end{vmatrix}
= 8
\]

(16)
\[
f(t) = \begin{vmatrix}
1 & 1 & 1 \\
a & b & t \\
a^2 & b^2 & t^2 \\
\end{vmatrix}
= \begin{bmatrix}
b & t \\
a & t^2 \\
a^2 & b^2 \\
\end{bmatrix}
= bt^2 - b^2t - at^2 + a^2t + ab^2 - a^2b
\]
\[
= (b - a)t^2 - (b^2 - a^2)t + ab(b - a) = (b - a)(t^2 - (a + b)t + ab) = (b - a)(t - a)(t - b)
\]

From the formulation above we see that \( f(a) = f(b) = 0 \). The matrix is invertible for all values of \( t \neq a, b \).

(21)
\[
\begin{vmatrix}
1 & 1 & 1 \\
x_1 & a_1 & b_1 \\
x_2 & a_2 & b_2 \\
\end{vmatrix}
= \begin{vmatrix}
a_1 & b_1 \\
al_2 & b_2 \\
x_1 & a_1 \\
x_2 & a_2 \\
\end{vmatrix}
= a_1 b_2 - a_2 b_1 - b_2 x_1 + b_1 x_2 + a_2 x_1 - a_1 x_2
\]
\[
= (a_2 - b_2)x_1 + (b_1 - a_1)x_2 + (a_1 b_2 - a_2 b_1) = 0
\]

This equation is equivalent to
\[
x_2 = \frac{a_2 - b_2}{a_1 - b_1} x_1 + \frac{a_1 b_2 - a_2 b_1}{a_1 - b_1}
\]

which is the equation of a line. There are several ways to see that the line must go through the points \( (a_1, a_2) \) and \( (b_1, b_2) \). One way is to substitute these points directly into the linear equation found above (as \( x_1 \) and \( x_2 \)) and verify that both sides of the equation are equal. Another way is to notice that substituting these points into the initial matrix as the values of \( x_1 \) and \( x_2 \) yields a matrix with two identical columns. If two of the columns are identical, they must be linearly dependent which means that the matrix is not invertible. Therefore, the determinant must be zero.

(31) We compute the Laplace expansion along the first column and notice that the only non-constant term will be the one corresponding to the fourth row (where the \( x \) is). Therefore,
the derivative of $f(x)$ will be the coefficient of $x$ which is:

$$
\begin{vmatrix}
1 & 2 & 3 & 4 \\
0 & 2 & 3 & 4 \\
0 & 0 & 3 & 4 \\
0 & 0 & 0 & 4 \\
\end{vmatrix} = -1 \cdot 2 \cdot 3 \cdot 4 = -24
$$

(32) By fact 6.2.4 we use an elementary column operation and see that

$$
\begin{vmatrix}
a & 3 & d \\
b & 3 & e \\
c & 3 & f \\
\end{vmatrix} = 3
\begin{vmatrix}
a & 1 & d \\
b & 1 & e \\
c & 1 & f \\
\end{vmatrix} = 3 \cdot 7 = 21
$$

By fact 6.2.3 we have

$$
\begin{vmatrix}
a & 3 & d \\
b & 5 & e \\
c & 7 & f \\
\end{vmatrix} = \begin{vmatrix} a & 1 & d \\
b & 1 & e \\
c & 1 & f \\
\end{vmatrix} + \begin{vmatrix} a & 2 & d \\
b & 4 & e \\
c & 6 & f \\
\end{vmatrix} + 2 \begin{vmatrix} a & 1 & d \\
b & 1 & e \\
c & 1 & f \\
\end{vmatrix} = 7 + 2(11) = 29
$$

(46) We must first find the matrix that corresponds to the linear transformation. We will use the standard basis $\{1, t, t^2\}$. $T(1) = 1$, $T(t) = 3t - 2$, $T(t^2) = (3t - 2)^2 = 9t^2 - 12t + 4$. The corresponding matrix is

$$
\begin{bmatrix}
1 & -2 & 4 \\
0 & 3 & -12 \\
0 & 0 & 9 \\
\end{bmatrix}
$$

Therefore, $\det(T) = \det(A) = 1(3)(9) = 27$. 