Section 5.3) 2, 3, 12, 13, 14, 15, 16, 17, 20, 22, 36, 38

(2,3) Let $A$ be the matrix for the orthogonal transformation $L$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then, by Fact 5.3.3 $A$ is orthogonal, and by Fact 5.3.7 $A^T A = I_n$. Therefore,

$$ L(\vec{v}) \cdot L(\vec{w}) = (A\vec{v}) \cdot (A\vec{w}) = (A\vec{v})^T (A\vec{w}) = \vec{v}^T A^T A \vec{w} = \vec{v}^T (A^T A) \vec{w} = \vec{v}^T \vec{w} = \vec{v} \cdot \vec{w} $$

Therefore, orthogonal transformations preserve the dot product. The angle between two nonzero vectors $\vec{v}$ and $\vec{w}$ is defined by the formula

$$ \cos \theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \ ||\vec{w}||} $$

Since $L$ is orthogonal, $||\vec{v}|| = ||L(\vec{v})||$ and $||\vec{w}|| = ||L(\vec{w})||$, so the angle between $L(\vec{v})$ and $L(\vec{w})$ is

$$ \cos \theta = \frac{L(\vec{v}) \cdot L(\vec{w})}{||L(\vec{v})|| \ ||L(\vec{w})||} = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \ ||\vec{w}||} $$

Therefore, orthogonal transformations preserve angle.

(12) We must find a unit vector that is perpendicular to both

\[
\begin{bmatrix}
  2/3 \\
  2/3 \\
  1/3
\end{bmatrix}
\quad \text{and} \quad 
\begin{bmatrix}
  1/\sqrt{2} \\
  -1/\sqrt{2} \\
  0
\end{bmatrix}
\]

One way to find a vector perpendicular to the given two is to take their cross product. Other methods do exist. For example, one could form a matrix using the given vectors as the rows and then find the kernel of the matrix. Here, I will use the cross product.

\[
\begin{bmatrix}
  2/3 \\
  2/3 \\
  1/3
\end{bmatrix} \times \begin{bmatrix}
  1/\sqrt{2} \\
  -1/\sqrt{2} \\
  0
\end{bmatrix} = \begin{bmatrix}
  0 + 1/3\sqrt{2} \\
  1/3\sqrt{2} - 0 \\
  -2/3\sqrt{2} - 2/3\sqrt{2}
\end{bmatrix} = \begin{bmatrix}
  1/3\sqrt{2} \\
  1/3\sqrt{2} \\
  -4/3\sqrt{2}
\end{bmatrix}
\]

Now that we have a vector that is orthogonal to the other two we must normalize it.

\[
\begin{vmatrix}
  1/3\sqrt{2} \\
  1/3\sqrt{2} \\
  -4/3\sqrt{2}
\end{vmatrix} = \sqrt{\frac{1}{18} + \frac{1}{18} + \frac{16}{18}} = 1
\]

The magnitude of the vector is already one. The desired matrix is:

\[
\begin{bmatrix}
  2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\
  2/3 & -1/\sqrt{2} & 1/3\sqrt{2} \\
  1/3 & 0 & -4/3\sqrt{2}
\end{bmatrix}
\]
(13) From problems 2 and 3, we know that orthogonal transformations preserve angles between vectors. Here we notice that \[
\begin{bmatrix}
2 \\
3 \\
0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
-3 \\
2 \\
0
\end{bmatrix}
\text{ are perpendicular, whereas }
T\left(\begin{bmatrix}
2 \\
3 \\
0
\end{bmatrix}\right) =
\begin{bmatrix}
3 \\
0 \\
2
\end{bmatrix}
\text{ and }
T\left(\begin{bmatrix}
-3 \\
2 \\
0
\end{bmatrix}\right) =
\begin{bmatrix}
2 \\
-3 \\
0
\end{bmatrix}
\text{ are not (look at the corresponding dot products). Therefore, there is no orthogonal transformation with the given criteria.}
\]

(14) By definition, a matrix is symmetric if it is equal to its transpose.
\[
(A^T A)^T = A^T (A^T)^T = A^T A
\]
\[
(A A^T)^T = (A^T)^T A^T = AA^T
\]
Both \(AA^T\) and \(A^T A\) are symmetric.

(15) We are given that \(A^T = A\) and \(B^T = B\). Therefore,
\[
(AB)^T = B^T A^T = BA
\]
This is only equal to \(AB\) if the matrices \(A\) and \(B\) commute with one another. In general, the product of symmetric matrices is not symmetric.

(16) We are given that \(A^T = A\). Therefore,
\[
(A^2)^T = (AA)^T = A^T A^T = AA = A^2
\]
The square of a symmetric matrix is always symmetric.

(17) We are given that \(A^T = A\). Therefore,
\[
(A^{-1})^T = (A^T)^{-1} = (A)^{-1} = A^{-1}
\]
The inverse of a symmetric matrix is always symmetric.
(20) First, we must find an orthonormal basis for the given subspace. Then, we will use Fact 5.3.10.

\[
\vec{w}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

\[
\vec{v}_2 \cdot \vec{w}_1 = \frac{1}{2} + \frac{9}{2} - \frac{5}{2} + \frac{3}{2} = 4
\]

\[
\vec{v}_2 - 4\vec{w}_1 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}
\]

\[
\vec{w}_2 = \frac{1}{10} \begin{bmatrix} -1 \\ 7 \\ -7 \\ 1 \end{bmatrix}
\]

\[
AA^T = \begin{bmatrix} 1/2 & -1/10 \\ 1/2 & 7/10 \\ 1/2 & -7/10 \\ 1/2 & 1/10 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/10 & 7/10 & -7/10 & 1/10 \end{bmatrix} = \frac{1}{50} \begin{bmatrix} 13 & 9 & 16 & 12 \\ 9 & 37 & -12 & 16 \\ 16 & -12 & 37 & 9 \\ 12 & 16 & 9 & 13 \end{bmatrix}
\]

(22) We can think of \(A^2\) as \(AA\). Then \(A(A\vec{x})\) first projects the vector \(\vec{x}\) orthogonally onto the subspace \(V\) defined by the columns of \(A\). Then, we perform the orthogonal projection again on the new vector \(\text{proj}_V \vec{x}\). But, \(\text{proj}_V \vec{x}\) is already in the subspace \(V\) so the orthogonal projection will return the same vector back. Therefore, \(A^2 = A\) if \(A\) is the matrix of an orthogonal projection.

Now, consider an orthonormal basis \(\{\vec{v}_1, \ldots, \vec{v}_m\}\) for the subspace \(V\) and define the matrix \(B = [\vec{v}_1 \cdots \vec{v}_m]\). Then, by Fact 5.3.10 the matrix of the orthogonal projection onto \(V\) is \(A = BB^T\). Therefore, \(A^2 = (BB^T)(BB^T) = B(B^T B)B^T\) since matrix multiplication is associative. Since the matrix \(B\) is orthogonal, we know that \(B^T B = I_m\). Therefore, \(A^2 = B(B^T B)B^T = BB^T = A\).
We are given \( L(A) = A^T \). Let \( A \) and \( B \) be 2 x 3 matrices, and let \( k \) be a scalar. Then,

\[
L(A + B) = (A + B)^T = A^T + B^T = L(A) + L(B)
\]

\[
L(kA) = (kA)^T = kA^T = kL(A)
\]

so \( L \) is a linear transformation. We see that \( L(L(A)) = L(A^T) = (A^T)^T = A \), so the transformation is its own inverse. Since \( L^{-1} \) exists, \( L \) is an isomorphism.

We will use the general matrix \( A \).

\[
T(A) = A + A^T = \begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} + \begin{bmatrix}
a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\
a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,n} & a_{2,n} & \cdots & a_{n,n}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
2a_{1,1} & a_{1,2} + a_{2,1} & \cdots & a_{1,n} + a_{n,1} \\
a_{1,2} + a_{2,1} & 2a_{2,2} & \cdots & a_{2,n} + a_{n,2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,n} + a_{n,1} & a_{2,n} + a_{n,2} & \cdots & 2a_{n,n}
\end{bmatrix}
\]

Looking at the calculations above, we see that all matrices in the image are symmetric. Therefore, the image of \( T \) is the set of all \( n \times n \) symmetric matrices. The kernel is the set of all matrices satisfying

\[
\begin{bmatrix}
2a_{1,1} & a_{1,2} + a_{2,1} & \cdots & a_{1,n} + a_{n,1} \\
a_{1,2} + a_{2,1} & 2a_{2,2} & \cdots & a_{2,n} + a_{n,2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1,n} + a_{n,1} & a_{2,n} + a_{n,2} & \cdots & 2a_{n,n}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

We see that the diagonal entries of \( A \) must equal zero. The off-diagonal entries must satisfy \( a_{i,j} = -a_{j,i} \) for all \( i \neq j \). Matrices that satisfy this criteria are called skew-symmetric, so the kernel of \( T \) is the set of all \( n \times n \) skew-symmetric matrices.