(1a) We use elementary row operations:

\[
\det(A) = \det \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 \\
2 & 2 & -3 & -3 \\
1 & -1 & -1 & -1
\end{bmatrix} = \det \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & -5 & -5 \\
0 & -2 & -2 & -2
\end{bmatrix} = 
\]

\[
= -\det \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & -2 & -2 & -2 \\
0 & 0 & -5 & -5 \\
0 & 0 & 0 & 1
\end{bmatrix} = -(1)(-2)(-5)(1) = -10
\]

(1b) \(\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = 100\)

(1c) \(\det(A + A) = \det(2A) = 2^4\det(A) = 16(-10) = -160\)

(1d) \(\det(A^{-1}) = \frac{1}{\det(A)} = -\frac{1}{10}\)

(2)

\[
T(e^x) = (e^x)'' - (e^x)' = e^x - e^x = 0 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
T(x) = (x)'' - (x)' = 0 - 1 = -1 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]

\[
B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}
\]

\[
T(1) = (1)'' - (1)' = 0 - 0 = 0 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

A basis for the kernel of \(T\) is \(\{e^x, 1\}\). A basis for the image of \(T\) is \(\{1\}\). \(T\) is not an isomorphism. There are several ways to see this. First, the kernel of \(T\) contains more than the zero element. Second, the image of \(T\) is not the entire space \(V\). Third, the \(B\)-matrix of \(T\) is not invertible.

(2e)

\[
< e^x, 1 > = \int_0^1 (e^x \cdot 1) \, dx = [e^x]_0^1 = e^1 - e^0 = e - 1 \neq 0
\]

The inner product of \(e^x\) and 1 is not zero, so they are not orthogonal.
(2f)  
\[< x - 1, x - 1 > = \int_{0}^{1} (x - 1)^2 \, dx = \int_{0}^{1} (x^2 - 2x + 1) \, dx = \left[ \frac{1}{3}x^3 - x^2 + x \right]_{0}^{1} = \frac{1}{3} - 1 + 1 = \frac{1}{3} \]

Therefore, the distance between \( x \) and 1 is \( \frac{1}{\sqrt{3}} \).

(3)  
\[\text{proj}_V p = < f, p > f + < g, p > g + < h, p > h = 2f - g + 3h\]

(4) We use Cramer’s Rule  
\[x_1 = \frac{\det(A_1)}{\det(A)} = \frac{4}{4} = 1, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{8}{4} = 2, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{12}{4} = 3\]

\[\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\]

(5a) FALSE: \( T(A) = A + A^T - I \) is not a linear transformation. Notice that \( T(A + B) = A + B + A^T + B^T - I = T(A) + T(B) + I \).


(5c) TRUE: If \( A \) and \( B \) are symmetric, but their product \( AB \) is not symmetric, then \( A \) and \( B \) must not commute with one another. Notice that \( (AB)^T = B^T A^T = BA \). If \( A \) and \( B \) commute, \( AB = BA \), so \( (AB)^T = AB \) implying that the product \( AB \) is symmetric. Therefore, \( A \) and \( B \) must not commute.

(5d) FALSE: The following is not an inner product on \( \mathbb{R}^4 \): \( < \vec{x}, \vec{y} > = x_1y_1 \). Notice that \( < \vec{e}_2, \vec{e}_2 > = 0 \) breaking the fourth property of inner products. (The inner product of a vector with itself is equal to zero if and only if it is the zero vector.)

(5e) FALSE: The following is not an orthonormal basis for \( \mathbb{R}^4 \) as the column vectors are not orthogonal to one another. However, they are all unit vectors.

\[
\begin{bmatrix}
1/2 & -1/2 & -1/2 & -1/2 \\
1/2 & 1/2 & -1/2 & -1/2 \\
1/2 & 1/2 & 1/2 & -1/2 \\
1/2 & 1/2 & 1/2 & 1/2 \\
\end{bmatrix}
\]
(5f) FALSE: The following matrix is not orthogonal as the column vectors are not unit vectors.

\[
\begin{bmatrix}
3 & -4 \\
4 & 3
\end{bmatrix}
\]

(5g) TRUE: If \(\{\vec{u}, \vec{v}, \vec{w}, \vec{z}\}\) is an orthonormal collection of vectors then \(||\vec{u} + \vec{v} + \vec{w} + \vec{z}|| = 2\).

We can use the Pythagorean Theorem to see that

\[
||\vec{u} + \vec{v} + \vec{w} + \vec{z}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 + ||\vec{w}||^2 + ||\vec{z}||^2 = 1 + 1 + 1 + 1 = 4
\]

\[
||\vec{u} + \vec{v} + \vec{w} + \vec{z}|| = \sqrt{4} = 2
\]

(5h) FALSE: If \(T\) is a linear transformation from \(V\) to \(W\), \(\dim(V) = 5\), \(\dim(W) = 7\) and \(\dim(\text{im}(T)) = 4\), the dimension of the orthogonal complement of \(\ker(T)\) is 4. By the rank-nullity theorem,

\[
\dim(\text{im}(T)) + \dim(\ker(T)) = \dim(V) \Rightarrow 4 + \dim(\ker(T)) = 5 \Rightarrow \dim(\ker(T)) = 1
\]

The sum of the dimension of a subspace and the dimension of its orthogonal complement is equal to the dimension of the entire space. The kernel of \(T\) is a subspace of \(V\). Therefore, the dimension of the orthogonal complement of \(\ker(T)\) must equal \(5 - 1 = 4\).

(6)

\[
||v + w||^2 - ||v - w||^2 = <v + w, v + w> - <v - w, v - w>
\]

\[
= <v, v> + <v, w> + <w, v> + <w, w> - <v, v> - <v, w> - <w, v> - <w, w>
\]

\[
= <v, v> + <v, w> + <w, w> - <v, v> + <w, w> - <v, w> + <w, w>
\]

\[
= 4 <v, w>
\]