These questions are correlated with the problems on exam 2. If you lost points for a given question on exam 2, you can gain half of the points back by completing (correctly) the corresponding question on the following list. For example, if you lost credit on problem 2c on the original exam, but got full credit for all other parts of problem 2, you only need to complete the extra problem 2c to get additional credit. An exception to this is if you are using a result of an earlier part to justify your answer to a later part. In this case, I expect to see the work that led to your justification. On problem 4, parts a & b have been combined, so you should complete the extra problem 4 to get credit back for either 4a or 4b.

You should complete the extra problems on your own and hand in your own work. However, you are free to use your book, class notes, homework, homework solutions, exams and exam solutions. I highly encourage you to look at the solutions to exam 2 before you complete the problems below. I will not be answering detailed questions related to these extra problems, but am very happy to go into detail answering questions related to the original exam 2. If you don’t understand anything on the original exam 2 please talk to me, others in the class, a tutor in the math center, or any other professor. You should notice that the topics of the problems are highly related so understanding exam 2 will make the extra problems much easier.

In order to receive full credit you must show all of your work and justify your conclusions. I expect your answers to be neat, organized and very easy to read. I reserve the right to not grade a paper that has lots of cross outs and arrows and is in general difficult to follow. Do your work on scrap paper first and then copy over your final answer.

Please hand in these problems and your original exam before 2pm on Friday April 9. You may hand in your work at class, or bring it to my office (LCB 333). If I’m not in my office, slide your paper under my office door and send me an email saying that you did so. I will not accept papers that arrive at my office after 2pm on Friday April 9.

(1) For the given matrix $A$, compute the specified quantities.

$$ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & -3 \end{bmatrix} $$

(a) $\det(A)$
(b) $\det(AA^T A)$
(c) $\det(A - 3A)$
(d) $\det(A^{-1})$
(2) Let $V$ be the inner product space of all solutions to the differential equation $f''' = f''$. It is known that $V$ is a three-dimensional linear space and one possible basis is $\{e^x, x, 1\}$. Let $T : V \to V$ be the linear transformation defined by $T(f) = f - f'$. Define the inner product on $V$ as

$$< f(x), g(x) > = \int_0^1 f(x)g(x)dx$$

(a) Find the matrix representation of $T$ with respect to the given basis.
(b) Find a basis for the kernel of $T$.
(c) Find a basis for the image of $T$.
(d) Is $T$ an isomorphism? Justify your answer.
(e) Are $(2x - 1)$ and 1 orthogonal on this inner product space?
(f) Find the distance between $(x + 1)$ and 1 for this inner product space.

(3) Assume that $W$ is an inner product space. Let $V$ be a subspace of $W$ with basis $\{f, g, h\}$. (Notice that the information only says basis, not a special kind of basis.) Use the following information to find the element of $V$ that is closest to $p$, where $p$ is an element of $W$:

$$< f, f >= 1 \quad < g, g >= 4 \quad < h, h >= 9$$
$$< f, g >= 0 \quad < f, h >= 0 \quad < g, h >= 0$$
$$< f, p >= 2 \quad < p, g >= -8 \quad < h, p >= 9$$

(4) (a & b combined) Using only the information given below, find $A^{-1}$ where $A$ is the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix}$$

No credit will be given for augmenting a matrix and row reducing.

$$\begin{align*}
\det \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} &= 5 \\
\det \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix} &= 1 \\
\det \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} &= 3 \\
\det \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} &= 5 \\
\det \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} &= -1 \\
\det \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} &= 7 \\
\det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} &= 1 \\
\det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} &= 1 \\
\det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} &= 3 \\
\det \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix} &= 4
\end{align*}$$
(5) Determine if the following statements are True or False. Answer True only if the statement is always true without any further criteria required. Write the word True or False. Do not write T or F. Justify your answer.

For the following questions, \( T \) is a transformation, and \( A \) and \( B \) are \( n \times n \) matrices.

(a) \( T(A) = AA^T \) is a linear transformation.

(b) The matrix \( (A^T A + A) \) is symmetric.

(c) If \( A \) and \( B \) are symmetric, and \( A \) and \( B \) commute with one another, then their product \( AB \) must be symmetric.

(d) The following is an inner product on \( \mathbb{R}^2 \): \( \langle \vec{x}, \vec{y} \rangle = x_1 y_1 + 2x_2 y_2 \).

(e) The following is an orthonormal basis for \( \mathbb{R}^4 \):

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix},
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix},
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

(f) The following matrix is orthogonal: \( \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \).

(g) Let \( \{\vec{v}, \vec{w}, \vec{x}, \vec{y}, \vec{z}\} \) be an orthonormal collection of vectors. Then \( ||\vec{v} + 2\vec{w} + 3\vec{x} + \vec{y} + \vec{z}|| = 4 \).

(h) Let \( T \) be a linear transformation from \( V \) to \( W \) where \( \text{dim}(V) = 6 \), \( \text{dim}(W) = 9 \) and the dimension of the orthogonal complement of \( \ker(T) \) is 2. Then, \( \text{dim}(\text{im}(T)) = 2 \).

(6) Let \( V \) be an inner product space. Using the definition of magnitude and properties of inner products, show that the following is true for all \( v \) and \( w \) in \( V \). (Notice that there is no special relationship between \( v \) and \( w \).) Show all steps.

\[
||v + w||^2 + ||v - w||^2 = 2||v||^2 + 2||w||^2
\]