(1)(16 points total) Let $A$ be the matrix

$$A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 \\
2 & 2 & -3 & -3 \\
1 & -1 & -1 & -1
\end{bmatrix}$$

Compute the following quantities. (Hint: Remember that there are several different ways to compute the determinant of a matrix and often one method is much simpler than another).

(a) (7 points) $\det(A)$
(b) (3 points) $\det(A^T A)$
(c) (3 points) $\det(A + A)$
(d) (3 points) $\det(A^{-1})$
Let $V$ be the inner product space of all solutions to the differential equation $f''' = f''$. It is known that $V$ is a three-dimensional linear space and one possible basis is $\{e^x, x, 1\}$. Let $T : V \to V$ be the linear transformation defined by $T(f) = f'' - f'$. Define the inner product on $V$ as
\[
<f(x), g(x)> = \int_0^1 f(x)g(x)dx
\]

(a) (10 points) Find the matrix representation of $T$ with respect to the given basis.

(b) (4 points) Find a basis for the kernel of $T$.

(c) (4 points) Find a basis for the image of $T$. 
(d) (4 points) Is $T$ an isomorphism? Justify your answer.

(e) (6 points) Are $e^x$ and 1 orthogonal on this inner product space?

(f) (6 points) Find the distance between $x$ and 1 for this inner product space.
(3) (4 points) Assume that W is an inner product space. Let V be a subspace of W with orthonormal basis \{f, g, h\}. For the element p of W, we have \langle f, p \rangle = 2, \langle p, g \rangle = -1 and \langle h, p \rangle = 3. Find the element of V that is closest to p.

(4) (6 points total)
(a) (4 points) Using only the information given below, solve the following linear system:

\[
\begin{bmatrix}
1 & 2 & 1 \\
-2 & 1 & 1 \\
-3 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
8 \\
3 \\
5
\end{bmatrix}
\]

No credit will be given for augmenting a vector and row reducing.

\[
\begin{align*}
\det\begin{bmatrix}
1 & 2 & 1 \\
-2 & 1 & 1 \\
8 & 3 & 5
\end{bmatrix} &= 24 \\
\det\begin{bmatrix}
8 & 3 & 5 \\
-2 & 1 & 1 \\
-3 & 1 & 2
\end{bmatrix} &= 16 \\
\det\begin{bmatrix}
1 & 8 & 1 \\
-2 & 3 & 1 \\
-3 & 5 & 2
\end{bmatrix} &= 8 \\
\det\begin{bmatrix}
1 & 2 & 1 \\
-2 & 1 & 1 \\
-3 & 1 & 2
\end{bmatrix} &= 4
\end{align*}
\]

(b) (2 points) What is the name of the method you used to get your answer to part (a)?
(5) (32 points total) Determine if the following statements are True or False. Answer True only if the statement is always true without any further criteria required. Write the word True or False. Do not write T or F. Justify your answer. (2 points for the right answer, 2 points for the justification on each part)

For the following questions, $T$ is a transformation, $A$ and $B$ are $n \times n$ matrices and $I$ is the $n \times n$ identity matrix.

(a) $T(A) = A + A^T - I$ is a linear transformation.

(b) The matrix $(A^T A + 4I)$ is symmetric.

(c) If $A$ and $B$ are symmetric, but their product $AB$ is not symmetric, then $A$ and $B$ must not commute with one another.

(d) The following is an inner product on $\mathbb{R}^4$: $\langle \vec{x}, \vec{y} \rangle = x_1 y_1$. 


(e) The following is an orthonormal basis for $\mathbb{R}^4$:
\[
\left\{\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}\right\}.
\]

(f) The following matrix is orthogonal:
\[
\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.
\]

(g) Assume that $\{\vec{u}, \vec{v}, \vec{w}, \vec{z}\}$ is an orthonormal collection of vectors. Then $||\vec{u} + \vec{v} + \vec{w} + \vec{z}|| = 2$.

(h) Let $T$ be a linear transformation from $V$ to $W$ where $\dim(V) = 5$, $\dim(W) = 7$ and $\dim(\text{im}(T)) = 4$. Then, the dimension of the orthogonal complement of $\ker(T)$ is 3.
(6) (8 points) Let $V$ be an inner product space. Using the definition of magnitude and properties of inner products, show that

$$||v + w||^2 - ||v - w||^2 = 4 < v, w >$$

for all $v$ and $w$ in $V$. Show all steps.