(1) 

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \]

Solutions to the linear system \( A\vec{x} = \vec{0} \) are of the form

\[ \vec{x} = r \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \]

where \( r \) is any scalar. The \( \dim(\ker(A))=1 \).

To find a basis for \( \text{im}(A) \) we extract the pivot columns of \( A \).

\[ \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \]

The \( \dim(\text{im}(A))=2 \).

(2a) The rank of a matrix \( A \) is the number of leading ones in \( \text{rref}(A) \).

(2b) We say that the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) form a basis of a subspace \( V \) of \( \mathbb{R}^m \) if they span \( V \) and are linearly independent.

(2c) Consider a subspace \( V \) of \( \mathbb{R}^5 \) with \( \dim(V)=3 \). We can find at most three linearly independent vectors in \( V \).

(2d) If \( A \) is a 6 x 5 matrix and \( \text{nullity}(A)=2 \), then the rank of \( A \) must be 3, by the Rank-Nullity Theorem.

(3a) FALSE: If \( V \) is a subspace of dimension 10, then any collection of more than 10 vectors does not necessarily span \( V \). For example, the collection of vectors \( \{\vec{v}, 2\vec{v}, 3\vec{v}, 4\vec{v}, 5\vec{v}, 6\vec{v}, 7\vec{v}, 8\vec{v}, 9\vec{v}, 10\vec{v}, 11\vec{v}\} \) where \( \vec{v} \in V \) spans a subspace of dimension 1 (a line). To span the subspace \( V \), a subset of 10 vectors from the collection must be linearly independent.

(3b) FALSE: If the columns of a 5 x 5 matrix \( A \) span \( \mathbb{R}^5 \), then the linear system \( A\vec{x} = \vec{0} \) has one unique solution.
(3c) TRUE: If $A$ is the coefficient matrix for a linear system of 6 equations with 4 unknowns, and $\text{rank}(A) = 4$, then the system has either one solution or none.

(3d) FALSE: Let $A$ be a 4 x 4 matrix. Consider a vector $\vec{b} \in \mathbb{R}^4$. If $A^{-1}$ does not exist, then the linear system $A\vec{x} = \vec{b}$ has either no solution or infinite solutions.

(3e) FALSE: There do not exist 2 x 2 invertible matrices $A$, $B$ and $C$ such that the product $BCA$ is not invertible. If $A$, $B$ and $C$ are invertible, then the matrix product $A^{-1}C^{-1}B^{-1}$ is the inverse of the product $BCA$.

(3f) FALSE: If $T$ is a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^7$ defined as $T(\vec{x}) = A\vec{x}$, then $\ker(A)$ is a subspace of $\mathbb{R}^5$.

(3g) TRUE: If the columns of an $n \times m$ matrix $A$ form a basis for the image of $A$, then $\ker(A) = \{\vec{0}\}$. If the columns of $A$ form a basis for its image, then the dimension of the image must be $m$. By the Rank-Nullity Theorem, the dimension of the kernel must be $m - \text{dim}(\text{im}(A)) = m - m = 0$.

(3h) TRUE: The only 3 dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself. If we take any 3 linearly independent vectors in $\mathbb{R}^3$, they must span all of $\mathbb{R}^3$.

(4a) A linear transformation $F$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called a shear parallel to $L$ if:

- $F(\vec{v}) = \vec{v}$ for all vectors $\vec{v}$ on $L$, and
- $F(\vec{x}) - \vec{x}$ is parallel to $L$ for all vectors $\vec{x} \in \mathbb{R}^2$.

(4b) We can use the determinant formula derived in class with $\det(S) = -1 - 1 = -2$.

$$ S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \quad \quad S^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} $$

We check that

$$ SS^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I_2 $$

(4c) We use the fact that $S[\vec{x}]_B = \vec{x}$ and $[\vec{x}]_B = S^{-1} \vec{x}$.

$$ [\vec{e}_1]_B = S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} $$
\[ \vec{e}_2_B = S^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

(4d) \[ B = S^{-1}AS = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]

(4e) The linear transformation \( T \) is not purely a shear. Instead, it is a shear along the line spanned by \[ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \] followed by a dilation by a factor of 2. We see this by looking at the \( B \)-matrix of the transformation \( T \). Let \( \vec{v}_1 \) and \( \vec{v}_2 \) represent the basis vectors of the basis \( B \), so \( [\vec{v}_1]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( [\vec{v}_2]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Then, \( T([\vec{v}_1]_B) = B[\vec{v}_1]_B = B\vec{e}_1 \) is the first column of the matrix \( B \), or \( 2\vec{e}_1 = 2[\vec{v}_1]_B \). So, the vector \( \vec{v}_1 \) is mapped to itself and then dilated by a factor of 2. For any vector \( [\vec{x}]_B \) in \( \mathbb{R}^2 \), we have

\[
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + b \\ b \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} = b \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

confirming that \( 1/2 \) times the matrix \( B \) produces a shear along the line spanned by \( \vec{v}_1 \). We follow this shear with a dilation by a factor of 2.