This exam is closed book and closed notes. You may not use a calculator of any kind. In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions. This exam is worth a total of 100 points.

(1) (18 points total) All parts of this problem refer to the matrix

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
-1 & -1 & -1 \\
0 & 1 & 2 \\
\end{bmatrix}
\]

(a) (10 points) Find all solutions, \( \vec{x} \), to the linear system \( A\vec{x} = \vec{0} \).
(b) (2 points) What is \( \dim(\ker(A)) \)?
(c) (4 points) Find a basis for \( \text{im}(A) \).
(d) (2 points) What is \( \dim(\text{im}(A)) \)?
(2)(14 points total) Answer the following questions.

(a)(4 points) What is the definition of the rank of a matrix $A$?

(b)(6 points) We say that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a basis of a subspace $V$ of $\mathbb{R}^m$ if what two criteria hold?

(c)(2 points) Consider a subspace $V$ of $\mathbb{R}^5$ with $\dim(V) = 3$. We can find at most how many linearly independent vectors in $V$?

(d)(2 points) If $A$ is a $6 \times 5$ matrix and $\text{nullity}(A) = 2$ what can we say about the rank of $A$?
(3) (32 points total) Determine if the following statements are True or False. Answer True only if the statement is always true without any further criteria required. Write the word True or False. Do not write T or F. Justify your answer. (2 points for the right answer, 2 points for the justification on each part)

(a) If $V$ is a subspace of dimension 10, then any collection of more than 10 vectors must span $V$.

(b) If the column vectors of a 5 x 5 matrix $A$ span $\mathbb{R}^5$, then the linear system $A\vec{x} = \vec{0}$ has infinitely many solutions.

(c) If $A$ is the coefficient matrix for a linear system of 6 equations with 4 unknowns, and $\text{rank}(A) = 4$, then the system has at most one solution.

(d) Let $A$ be a 4 x 4 matrix. Consider a vector $\vec{b} \in \mathbb{R}^4$. If $A^{-1}$ does not exist, then the linear system $A\vec{x} = \vec{b}$ is inconsistent.
(e) There exist 2 x 2 invertible matrices $A$, $B$ and $C$ such that the product $BCA$ is not invertible.

(f) If $T$ is a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^7$ defined as $T(\vec{x}) = A\vec{x}$, then $\ker(A)$ is a subspace of $\mathbb{R}^7$.

(g) If the columns of an $n \times m$ matrix $A$ form a basis for the image of $A$, then $\ker(A) = \{\vec{0}\}$.

(h) The only 3 dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.
(4) (36 points total) For all parts of problem (4) assume the following.

$T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

$B$ is the following basis for $\mathbb{R}^2$:

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$L$ is a line in $\mathbb{R}^2$ spanned by the vector $\vec{w} \in \mathbb{R}^2$

(a) (6 points) A linear transformation $F$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called a shear parallel to $L$ if what two properties hold?

(b) (8 points) Let $S$ be the matrix whose columns are the basis vectors of $B$. Find $S^{-1}$ using any method discussed in class. Check your answer by computing the product $SS^{-1}$.
(c) (6 points) Find the $B$-coordinate vectors for $\vec{e}_1$ and $\vec{e}_2$ (the standard basis vectors in $\mathbb{R}^2$).

(d) (10 points) Find the $B$-matrix of $T$. Label it $B$. 
(e) (6 points) Interpret the transformation $T$ geometrically by answering one of the following sets of questions. You should use information from parts (a)-(d) of this problem. It may help to draw a picture, but this is not required and will not substitute for an algebraic proof.

(i) Is the linear transformation $T$ a pure shear? If so, prove it by verifying the two properties you listed in part (a) and find the line $L$ that the shear is parallel to.

(ii) If the linear transformation $T$ is not a shear, prove it by showing that one of the criteria listed in part (a) fails. Is $T$ the composite of a shear with something else? Explain how you reached your conclusion.