Introduction to Factoring

The goal of factoring is to take a quadratic equation in standard form,

\[ ax^2 + bx + c = 0 \]

and rewrite it in factored form

\[ (x - r_1)(x - r_2) = 0 \]

Once our equation is in this form we can solve for \( x \) by remembering:

if \( A \cdot B = 0 \), then either \( A = 0 \), \( B = 0 \) or \( A = B = 0 \).

Thus, if we have \( (x - r_1)(x - r_2) = 0 \) we can find the \( x \)’s that solve our equation by solving:

\[ x - r_1 = 0 \Rightarrow x = r_1 \]
\[ x - r_2 = 0 \Rightarrow x = r_2 \]
Factoring: Three Cases

Case 1: Leading Coefficient is One

Example: \(x^2 - 11x + 24 = 0\)

Compare this equation to the standard quadratic form: \(a = 1, b = -11\) and \(c = 24\).

Question to ask yourself: “What two numbers multiply to equal \(c\) and add to equal \(b\)?”

So in our case, “What two numbers multiply to equal 24 but add to equal 11? If you are able to find these numbers easily in your head that is fine - if not, you can make a table like this:

<table>
<thead>
<tr>
<th></th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>(= (1)(24))</td>
<td>25</td>
</tr>
<tr>
<td>(= (-1)(-24))</td>
<td>-25</td>
</tr>
<tr>
<td>(= (2)(12))</td>
<td>14</td>
</tr>
<tr>
<td>(= (-2)(-12))</td>
<td>-14</td>
</tr>
<tr>
<td>(= (3)(8))</td>
<td>11</td>
</tr>
<tr>
<td>(= (-3)(-8))</td>
<td>-11</td>
</tr>
<tr>
<td>(= (4)(6))</td>
<td>10</td>
</tr>
<tr>
<td>(= (-4)(-6))</td>
<td>-10</td>
</tr>
</tbody>
</table>

We choose the pair \(-3, -8\).

Remember we were given: \(x^2 - 11x + 24 = 0\). The factored form is: \((x - 3)(x - 8) = 0\).

To find the \(x\)’s we solve the equations:

\[ x - 3 = 0 \Rightarrow x = 3 \]
\[ x - 8 = 0 \Rightarrow x = 8 \]
Case 2: Leading Coefficient can be Factored Out

Example: $3x^2 - 33x + 72 = 0$

Notice:

$$3x^2 - 33x + 72 = 3[x^2 - 11x + 24] = 0$$

Important! The “3” doesn’t disappear; it is simply “factored out”.

Now we can factor the portion of the expression in brackets exactly the way we did for case 1 to get:

$$3(x - 3)(x - 8) = 0$$

Clearly $3 \neq 0$, thus to find the $x$’s that solve our equation we solve the same pair of equations as in Case 1:

$$x - 3 = 0 \Rightarrow x = 3$$
$$x - 8 = 0 \Rightarrow x = 8$$

Case 3: Leading Coefficient Cannot be Factored Out

In this case the leading coefficient is not 1, but cannot be factored out without leaving fractions behind.

Ex: $6x^2 + 5x - 6 = 0$

Notice, if we try to factor out the 6 we get:

$$6[x^2 + \frac{5}{6}x - 1] = 0$$

Trying to factor quadratic equations containing fractions is a pain, so instead we will use a new method.

Step 1 - Number Choice: We begin by comparing this example to the standard form equation: $a = 6$, $b = 5$, and $c = -1$.

Question to ask yourself: “What two numbers multiply to equal $a \cdot c$ and add to equal $b$?
In our case: “What two numbers multiply to equal $(6)(-6) = -36$ and add to equal 5?"

If you are able to find these numbers easily in your head that is fine - if not you can make a chart like the one below:
<table>
<thead>
<tr>
<th>-6</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)(36)</td>
<td>35</td>
</tr>
<tr>
<td>(1)(-36)</td>
<td>-35</td>
</tr>
<tr>
<td>(-2)(18)</td>
<td>16</td>
</tr>
<tr>
<td>(2)(-18)</td>
<td>-16</td>
</tr>
<tr>
<td>(-3)(12)</td>
<td>9</td>
</tr>
<tr>
<td>(3)(-12)</td>
<td>-9</td>
</tr>
<tr>
<td>(-4)(9)</td>
<td>5</td>
</tr>
<tr>
<td>(4)(-9)</td>
<td>-5</td>
</tr>
<tr>
<td>(6)(-6)</td>
<td>0</td>
</tr>
</tbody>
</table>

**Step 2 - Rewrite your Equation:**

Notice: $5x = 9x - 4x$ Rewritten Equation: $6x^2 + 9x - 4x - 6 = 0$

**Step 3 - Group the Terms:**

$$(6x^2 + 9x) + (-4x - 6) = 0$$

Important: *Always* place an addition symbol between the two groups - this is still an addition/subtraction problem all you’ve done is group terms in parenthesis.

**Step 4 - Common Factors:**

Questions to ask yourself: (a) “What do the terms in my first group have in common?”: In our case: $3x$

(b) “What do the terms in my second group have in common?”: In our case: $-2$

**Step 5 - Rewrite:**

$$(6x^2 + 9x) + (-4x - 6) = 0$$

- is equivalent to -

$$3x(x + 3) - 2(2x + 3) = 0$$
Step 6 - Apply the “Opposite” of the Distributive Property:
Notice: The terms in the parenthesis match! (If it doesn’t you did something wrong)

\[3x(2x + 3) - 2(2x + 3) = 0 \Rightarrow (3x - 2)(x + 3) = 0\]

Step 7 - Solve:

\[3x - 2 = 0 \Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3}\]
\[2x + 3 = 0 \Rightarrow x = -\frac{3}{2}\]

Now you know how to solve equations by factoring!