When we first learned how to evaluate definite integrals of the form
\[ \int_{a}^{b} f(x) \, dx \]
it was assumed that the limits of integration, \( a \) and \( b \), are constants. However, in many physical problems, it makes perfect sense to talk about integrals with infinite limits. These integrals are called improper integrals.

Definition 1
\[ \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \]
\[ \int_{a}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx \]

If the limits on the right exist and have finite values, then we say that the corresponding improper integrals converge and have those values. Otherwise, the integrals are said to diverge.

Example 1 Determine
\[ \int_{0}^{\infty} e^{-x} \, dx \]
Solution We evaluate this integral according to the definition just given.
\[ \int_{0}^{\infty} e^{-x} \, dx = \lim_{b \to \infty} \int_{0}^{b} e^{-x} \, dx = \lim_{b \to \infty} \left[ -e^{-x} \right]_{0}^{b} = \lim_{b \to \infty} (1 - e^{-b}) = 1 \]

Example 2 Evaluate, if possible,
\[ \int_{1}^{\infty} \frac{dt}{t} \]
Solution As in the previous problem, we have
\[ \int_{0}^{\infty} \frac{dt}{t} = \lim_{b \to \infty} \int_{0}^{b} \frac{dt}{t} = \lim_{b \to \infty} [\ln t]_{0}^{b} = \lim_{b \to \infty} \ln b = \infty \]

This integral does not exist because it diverges.

Up to this point we have only considered integrals with one infinite limit. We define integrals where both limits are infinite in the following way:
Definition 2 If both \( \int_{-\infty}^{0} f(x) \, dx \) and \( \int_{0}^{\infty} f(x) \, dx \) converge, then \( \int_{-\infty}^{\infty} f(x) \, dx \) is said to converge and have value

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{\infty} f(x) \, dx
\]

Otherwise, \( \int_{-\infty}^{\infty} f(x) \, dx \) diverges.

Example 3 Calculate

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}
\]

Solution

\[
\int_{0}^{\infty} \frac{dx}{1 + x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1 + x^2} = \lim_{b \to \infty} \left[ \tan^{-1} x \right]_{0}^{b} = \lim_{b \to \infty} \frac{\pi}{2}
\]

Since the integrand is an even function,

\[
\int_{-\infty}^{0} \frac{dx}{1 + x^2} = \frac{\pi}{2}
\]

as well. Hence,

\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{0} \frac{dx}{1 + x^2} + \int_{0}^{\infty} \frac{dx}{1 + x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi
\]

Results associated with improper integrals often go against our best intuition. A classic example of this is known as the Paradox of Gabriel’s Horn. Let the curve \( y = \frac{1}{x} \) on \([1, \infty)\) be revolved about the \( x \)-axis, thereby generating a surface of revolution known as Gabriel’s Horn. The paradox in this problem results from the fact that the volume of the horn is finite, while the surface area is infinite. In practical terms, this means we could fill the horn with a finite amount of paint, but there would never be enough paint in the horn to paint its surface. We prove this in the following way:

\[
V = \int_{1}^{\infty} \pi \left( \frac{1}{x} \right)^2 \, dx = \lim_{b \to \infty} \pi \int_{1}^{b} x^{-2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_{1}^{b} = \pi
\]

\[
A = \int_{1}^{\infty} 2\pi y \, ds = \int_{1}^{\infty} 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 2\pi \int_{1}^{\infty} \frac{\sqrt{x^4 + 1}}{x^3} \, dx
\]

Now,

\[
\frac{\sqrt{x^4 + 1}}{x^3} > \frac{\sqrt{x^2}}{x^3} = \frac{1}{x}
\]

Thus,

\[
\int_{1}^{b} \frac{\sqrt{x^4 + 1}}{x^3} \, dx > \int_{1}^{b} \frac{1}{x} \, dx = \ln b
\]

Since \( \ln b \to \infty \) as \( b \to \infty \), we conclude that \( A \) is infinite.

Example 4 Show that \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \) diverges for \( p \leq 1 \) and converges for \( p > 1 \).
Solution  We have already shown that this integral diverges for $p = 1$. If $p \neq 1$,

$$\int_1^\infty \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_1^b x^{-p} \, dx = \lim_{b \to \infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^b = \lim_{b \to \infty} \left[ \frac{1}{1-p} \right] \left[ \frac{1}{b^{p-1}} - 1 \right]$$

The last expression is $\infty$ if $p < 1$ and $1/(p - 1)$ is $p > 1$. The conclusion follows.