At the beginning of 1998, the world’s population was about 5.9 billion. It is predicted that by the year 2020, the populations will reach 7.9 billion. How are such predictions made?

To treat the problem mathematically, let \( y = f(t) \) denote the size of the population at time \( t \), where \( t \) is the number of years since 1998. Actually, \( f(t) \) is an integer, and its graph “jumps” when someone is born or dies. However, for a large population, these jumps are so small relative to the total population that we will not go far wrong if we pretend that \( f \) is a nice differentiable function.

It seems reasonable to suppose that the increase \( \Delta y \) in population during a short time period \( \Delta t \) is proportional to the size of the population at the beginning of the period and to the length of the period. Thus, \( \Delta y = ky\Delta t \), or

\[
\frac{\Delta y}{\Delta t} = ky
\]

In its limiting form, this gives the differential equation

\[
\frac{dy}{dt} = ky
\]

If \( k > 0 \), the population is increasing; if \( k < 0 \), it is decreasing. For world population, history indicates that \( k \) is about 0.0132 (assuming \( t \) is measured in years).

Example 1 How long will it take the world population to double, under the assumptions about the population model given above?
Solution We need to solve the following for \( t \):

\[
11.8 = 5.9e^{0.0132t} \Rightarrow 2 = e^{0.0132t}
\]

since \( 11.8 = 2(5.9) \) and at time \( t = 0 \) the population is 5.9 billion. Taking logarithms of both sides gives

\[
\ln 2 = 0.0132t \Rightarrow t = \frac{\ln 2}{0.0132} \approx 53 \text{ years}
\]

If an exponentially growing quantity doubles from \( y_0 \) to \( 2y_0 \) in an interval of time \( T \), it will double in any interval of length \( T \), since

\[
\frac{y(t + T)}{y(t)} = \frac{y_0 e^{k(t+T)}}{y_0 e^{kt}} = \frac{y_0 e^{kT}}{y_0} = \frac{2y_0}{y_0} = 2
\]

We call the number \( T \) the doubling time.

The exponential model \( y = y_0 e^{kt}, k > 0 \), for population growth is flawed since it projects faster and faster growth indefinitely far into the future. In most cases, a limited amount of resources will eventually force a slowing of the growth rate. Therefore, it is generally assumed that the logistic model, in which we assume that the rate of growth is proportional both to the population size \( y \) and to the difference \( L - y \), where \( L \) is the carrying capacity of the system. This leads to the differential equation

\[
\frac{dy}{dt} = ky(L - y)
\]

Note that for small \( y \), \( \frac{dy}{dt} \approx kLy \), which suggest exponential type growth. But as \( y \) nears \( L \), growth is curtailed and \( \frac{dy}{dt} \) gets smaller and smaller, so that the solution asymptotically approaches the constant solution \( y = L \).

Not everything grows; some things decrease over time. For example, radioactive elements decay, and they do it at a rate proportional to the amount present. Thus, their changes also satisfy the differential equation

\[
\frac{dy}{dt} = ky
\]

where \( k < 0 \). It is still true that \( y = y_0 e^{kt} \) is the solution to this equation.

Example 2 Carbon 14, an isotope of carbon, is radioactive and decays at a rate proportional to the amount present. It’s half-life is 5730 years; that is, it takes 5730 years for a given amount of carbon 14 to decay to one-half its original size. If 10 grams was present originally, how much will be left after 2000 years.

Solution In order to solve this problem, we need to first determine \( k \), and then use the solution of the differential equation to determine the value of \( y \) at time \( t = 2000 \). The half-life of 5730 years gives us that

\[
5 = 10e^{k(5730)} \Rightarrow \frac{1}{2} = e^{k(5730)}
\]

or, after taking logarithms,

\[
-\ln 2 = 5730k \Rightarrow k = -\frac{\ln 2}{5730} \approx -0.000121
\]

Thus,

\[
y = 10e^{-0.000121t}
\]

and at time \( t = 2000 \) this gives

\[
y = 10e^{-0.000121(2000)} \approx 7.85 \text{ grams}
\]
If we put \( A_0 \) dollars in the bank at 100\( r \) percent compounded \( n \) times per year, it will be worth \( A(t) \) dollars at the end of \( t \) years, where
\[
A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}
\]

**Example 3** Suppose that you put $500 in the bank at 4\% interest, compounded monthly. How much will it be worth at the end of 3 years?

**Solution** Here \( r = 0.04 \) and \( n = 12 \), so
\[
A = 500 \left(1 + \frac{0.04}{12}\right)^{12 \times 3} \approx \$563.63
\]

Now let us consider what happens when interest is compounded continuously, that is, when \( n \), the number of compounding periods tends to infinity. In order to evaluate this case, we will need the following theorem.

**Theorem 1**
\[
\lim_{h \to 0} (1 + h)^{1/h} = e
\]

**Proof** Since \( f(x) = \ln x \Rightarrow f'(x) = 1 \), and in particular, \( f'(1) = 1 \), then from the definition of the derivative we get
\[
1 = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h \to 0} \frac{1}{h} \ln(1+h) = \lim_{h \to 0} \ln (1 + h)^{1/h}
\]

Now \( g(x) = e^x = \exp x \) is a continuous function, and therefore we can pass the limit inside the exponential function in the following argument:
\[
\lim_{h \to 0} (1 + h)^{1/h} = \lim_{h \to 0} \exp \left[ \ln (1 + h)^{1/h} \right] = \exp \left[ \lim_{h \to 0} \ln (1 + h)^{1/h} \right] = \exp 1 = e
\]

Therefore, for continuously compounded interest, we get
\[
A(t) = \lim_{n \to \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = A_0 \lim_{n \to \infty} \left[ \left(1 + \frac{r}{n}\right)^{n/r} \right]^{rt} = A_0 \left[ \lim_{h \to 0} (1 + h)^{1/h} \right]^{rt} = A_0 e^{rt}
\]