Subadditivity and Symbolic Powers

JMM 2018: AMS Special Session on Commutative Algebra in All Characteristics

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Motivation: Symbolic Powers

- Symbolic powers: $p^{(n)} := p^n R_p \cap R$. 

Evidently, $p^n \subseteq p^{(n)}$. Question: how far is this from equality?

Answer [Swanson, '00]: for nice rings, $\forall p \exists h \forall n: p^{(hn)} \subseteq p^n$.

Answer [ELS '01, HH '02, Hara '05, MS '17]: If $R$ is regular, then $h = \text{dim} R$ works for all $p$!

Question: can we find a uniform $h$ that works for non-regular rings?
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- **Question**: can we find a uniform \( h \) that works for non-regular rings?
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Sketch of ELS/Hara/MS proof: set \( d = \dim R \).

\[
\mathfrak{p}^{(dn)} \subseteq \tau(\mathfrak{p}^{(dn)}) = \tau\left(\left(\mathfrak{p}^{(dn)}\right)^{n/n}\right) \subseteq \tau\left(\left(\mathfrak{p}^{(dn)}\right)^{1/n}\right)^n \subseteq \mathfrak{p}^n
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(2): “Subadditivity”: need \( R \) regular! 
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(2): “Subadditivity”: need $R$ regular!
(3): Holds generally (theory of integral closures + Skoda’s theorem)
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Key idea: replace $\tau \left( p^{(dn)} \right)$ with an ideal so that (2) holds always, and hope (1) holds sometimes
$R$ a f.g. $k$-algebra, domain, char($k$) = $p > 0$.

- Frobenius: $F^e(x) = x^p$
Interlude: Cartier Algebras and Test Ideas

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$$r \cdot F^e_* x = F^e_* r^p x$$
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- Concretely, $F^e_\ast R = \{ F^e_\ast x \mid x \in R \} \cong R$ as ab. group, but $R$-module structure given by
  \[ r \cdot F^e_\ast x = F^e_\ast r^{p^e} x \]
- An $R$-linear map $\varphi : F^e_\ast R \to R$ satisfies
  \[ \varphi(F^e_\ast (a + b)) = \varphi(F^e_\ast a) + \varphi(F^e_\ast b), \]
  \[ \varphi(F^e_\ast r^{p^e} x) = \varphi(r F^e_\ast x) = r \varphi(F^e_\ast x) \]
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- $C_R := \bigcup_e \text{Hom}_R(F^e_* R, R)$.

- A Cartier algebra on $R$ is a subset $D \subseteq C_R$.

- $D_e := D \cap \text{Hom}_R(F^e_* R, R)$.

- Test ideal of $D$: $\tau(R, D) :=$ the unique, minimal $J \neq 0$ such that $\phi(F^e_* J) \subseteq J$ for all $e$, $\phi \in D_e$.

  (It exists!)
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- $\mathcal{C}_R := \bigcup_e \text{Hom}_R(F^e_* R, R)$.
- A Cartier algebra on $R$ is a subset $\mathcal{D} \subseteq \mathcal{C}_R$. 

\[(\text{It exists!})\]

"$J, \phi$ are compatible"
• \( C_R := \bigcup_e \text{Hom}_R(F^e_\ast R, R). \)

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Interlude: Cartier Algebras and Test Ideas

Multiplying a Cart. Alg. by an ideal

- Given $\mathcal{D}$, $\alpha_i \subseteq R$, construct

$$\mathcal{D}\alpha_1 \cdots \alpha_n := \bigcup_{e} \left\{ \varphi \left( F_e^* x \cdot - \right) \mid \varphi \in \mathcal{D}_e, x \in \prod_i \alpha_i^{p_e-1} \right\}$$
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- $\tau (R, a_1 \cdots a_n) := \tau (R, C_R a_1 \cdots a_n)$
The diagonal cartier algebra

- Subadditivity: if $R$ regular, then $\tau(R, ab) \subseteq \tau(R, a)\tau(R, b)$.
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- Answer 2: Use Cartier algebras!
- For any $n \in \mathbb{N}$, define $\mathcal{D}(n)_e$ as the set of $\varphi : F_\ast^e R \rightarrow R$ such that

\[
F_\ast^e R \otimes n \xrightarrow{\exists \varphi} R \otimes n
\]

\[
F_\ast^e \mu \downarrow \quad \mu
\]

\[
F_\ast^e R \xrightarrow{\varphi} R
\]
The diagonal cartier algebra

Theorem (S.)

\[ \tau(R, \mathcal{D}(n) a_1 \cdots a_n) \subseteq \tau(R, a_1) \cdots \tau(R, a_n) \]

Proof.
The diagonal cartier algebra

Theorem (S.)

\[ \tau(R, \mathcal{D}(n)a_1 \cdots a_n) \subseteq \tau(R, a_1) \cdots \tau(R, a_n) \]

Proof.

\[ \text{Hom}_{R \otimes n}(F_e^e R \otimes^n, R \otimes^n) \cong \text{Hom}_R(F_e^e R, R)^{\otimes n} \]

\[ \Rightarrow \tau(R^{\otimes n}, a_1 \otimes \cdots \otimes a_n) \subseteq \tau(R, a_1) \otimes \cdots \otimes \tau(R, a_n) \]
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The set \( \mathcal{D}(n) \) is constructed specifically so that

\[ \tau(R, \mathcal{D}(n)a_1 \cdots a_n) \subseteq \mu(\tau(R \otimes^n, a_1 \otimes \cdots \otimes a_n)) \]
Question: When is $p^{(dn)} \subseteq \tau(R, \mathcal{D}(n)p^{(dn)})$?

Just need $\mathcal{D}(n)$ to be large ($F$-regular)

Def: $R$ is diagonally $F$-regular if $\mathcal{D}(n)$ is $F$-regular for all $n$.

Question: Are there any non-regular rings that are diagonally $F$-regular?

Theorem (Carvajal-Rojas, S.)
Let $k$ be a field of characteristic $p$. Then the Segre product $k[x_0, \ldots, x_r] \# k[y_0, \ldots, y_s]$ is diagonally $F$-regular. Thus, $p^{(r+s+1)n} \subseteq p$ for all $p \in \text{Spec}(k[x_0, \ldots, x_r] \# k[y_0, \ldots, y_s])$. 
Upshot: Symbolic Powers and Diagonal $F$-regularity

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Computing $D(n)$

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- Affine toric $R \subseteq k[x_1, \ldots, x_d] \leftrightarrow$ cones $C \subseteq \mathbb{R}^d$
- $a \in \frac{1}{p^e} \mathbb{Z}^d \cap P_R \leftrightarrow$ gens $\pi_a \in \text{Hom}_R(F_e^* R, R)$
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**Theorem (S.)**

*If $R$ affine toric, then $\mathcal{D}(2)_e$ is generated by*

$$\{\pi_a \mid P_R \cap (a - P_R) \text{ is "big"} \}$$

$\mathcal{Z} \subseteq \mathbb{R}^d$ is big if $\forall v \in \frac{1}{p^e}\mathbb{Z}^d \exists s \in \mathbb{Z} : v + s \in \mathcal{Z}$
Example

\[ R = k[x, y, u, v]/(xy - uv) \cong k[x, y, u, xyu^{-1}] \]