Symbolic powers in rings of positive characteristic

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1 Learning objectives

1. We can understand the symbolic powers of (positive characteristic) rings by closely studying the maps $R^{1/p^n} \rightarrow R$.

2. For certain rings (e.g. Toric varieties, Hibi rings) the study of these maps boils down to (hard!) combinatorics.

2 Symbolic and ordinary powers of ideals

We assume, for simplicity:

Global assumptions: $R$ is a normal domain finitely generated over a perfect field $k$.

Though everything works even if $k$ is not perfect and $R$ is just reduced.

3. Definition: if $p \in \text{Spec } R$, we define $p^{(n)} := \ldots$

4. Remark: these are larger than ordinary powers, i.e. $p^{(n)} \supseteq p^n$. Rarely an equality.

5. Exercise: let $R = k[x, y, z]/(xy - z^5)$ and $p = (x, z)$. Then $p^{(5)} = \ldots \supseteq p^5$.

6. Exercise: Let $m \subseteq R$ be a maximal ideal. Then $m^{(n)} = m^n$ for all $n$.

7. Intuition: $p^{(n)}$ is the set of regular functions on Spec $R$ that \ldots (cf. Zariski-Nagata theorem).

Main question: How does $p^{(n)}$ relate to $p^n$? More precisely, for which $a, b \in \mathbb{N}$ do we have \ldots ?

8. In 2000, Ein, Lazarsfeld, and Smith gave a striking answer to this question:

Theorem 1 ([ELS01]). Let $R$ be a regular ring over an algebraically closed field of characteristic 0. Then $p^{(hn)} \subseteq p^n$ for all prime ideals $p$ of height $h$.

9. We will talk about weakening the regularity assumption in this theorem.

10. Remark: in particular, if $\dim R = d$, we see that $p^{(dn)} \subseteq p^n$ for all $p$ and all $n$. Because this number $d$ depends only on the ring $R$ (and not on the primes $p$) we say these rings have the Uniform Symbolic Topology Property, or USTP for short.
3 Commutative algebra mod $p$

11. To prove something like Theorem 1, it actually suffices to work with rings of positive characteristic, using standard “reduction mod $p$” techniques. For instance, to show that

$$R = \frac{\mathbb{C}[x, y, z]}{(x^3 - 5y^2 + 7z^3)}$$

has USTP it suffices to show that its reductions mod $p$,

$$R_p = \frac{\mathbb{C}[x, y, z]}{(x^3 - 5y^2 + 7z^3)}$$

have USTP for all $p \gg 0$. In general, there’s a rich theory saying that many properties of a ring in characteristic 0 can be checked mod $p$ sufficiently large. See [HH99, Chapter 2] for details.

12. **Exercise:** How would one define the reduction of a ring such as

$$S = \frac{\mathbb{C}[x, y, z]}{(\sqrt{2}x^3 - \pi y^2 + \frac{i}{7}z^3)}$$

modulo $p$?

13. **Now let** $R$ **have characteristic** $p > 0$. Consider the $R$-module, $R^{1/p}$ defined by $R^{1/p} :=$

**Key idea:** We can learn a lot about $R$ by studying the $R$-module structure of $R^{1/p^e}$ for $e > 0$

Note that $R^{1/p^e}$ is always a finitely generated module in our setting.

14. For instance, a theorem of Kunz says that $R$ is regular if and only if $R^{1/p^e}$ is a flat $R$-module for some (all) $e > 0$ [Kun69].

15. Example from number theory: these modules can be used to detect whether an elliptic curve in positive characteristic is “ordinary” or “supersingular” [BS15].

16. Recall: our goal is to weaken the regularity hypothesis in Theorem 1. The crux of Ein–Lazarsfeld–Smith’s proof$^1$ is the following chain of containments:

$$p^{(hn)} \subseteq \sum_{e > 0} \sum_{\varphi : R^{1/p^e} \to R} \varphi\left((p^{(hn)})^{1/p^e}\right) \subseteq \sum_{e > 0} \sum_{\varphi : R^{1/p^e} \to R} \varphi\left((p^{(hn)})^{[p^e/n]/p^e}\right)^n \subseteq p^n$$

The second containment breaks if $R$ is not regular! So we make the sum on the left a little smaller:

**Theorem 2 ([Smo18]).** Let $R$ be a normal domain finitely generated over a perfect field $k$ of characteristic $p$. Then, for all ideals $\mathfrak{a}$ of $R$, we have$^2$

$$\sum_{e > 0} \sum_{\varphi \in \mathfrak{a}^{1/p^e}(R)} \varphi\left(\mathfrak{a}^{1/p^e}\right) \subseteq \sum_{e > 0} \sum_{\varphi : R^{1/p^e} \to R} \varphi\left(\mathfrak{a}^{[p^e/n]/p^e}\right)^n,$$

$^1$At least, the positive-characteristic analog of their proof. The original proof uses *multiplier ideals* which are, fascinatingly, a close analog of these test ideals that works in characteristic 0. Constructing multiplier ideals requires resolution of singularities, which is not known in positive characteristic.

$^2$For the experts: I’m sacrificing precision for clarity by omitting test elements in the sums below.
where \( \mathcal{D}_e^{(n)}(R) \subseteq \text{Hom}_R(R^{1/p^e}, R) \) is the set of maps admitting a lifting to the \( n \)-fold tensor product:

\[
\begin{array}{ccc}
\left(R^\otimes k^n\right)^{1/p^e} & \longrightarrow & R^\otimes k^n \\
\downarrow & & \downarrow \\
R^{1/p^e} & \overset{\varphi}{\longrightarrow} & R
\end{array}
\]

17. I won’t explain how this works in this talk, but here’s the key take-away I want you to have from this discussion:

| Key idea: | This set of maps \( \mathcal{D}_e^{(n)}(R) \) is a correction term that accounts for our ring \( R \) not being regular. If the correction term is not too bad, then the conclusion of Theorem 1 still holds. [CS18, Theorem 4.1] |

18. **Definition:** If \( \mathcal{D}_e^{(n)}(R) \) is big enough for the argument to work (for some \( e \)), then \( R \) is called \( n \)-Diagonally \( F \)-Regular (\( n \)-DFR). If this is true for all \( n > 0 \), we say \( R \) is Diagonally \( F \)-Regular (DFR).

19. **Aside for experts:** Concretely, we need the test ideal of \( \mathcal{D}_e^{(n)}(R) \) to be all of \( R \), i.e.

\[
\sum_e \sum_{\varphi \in \mathcal{D}_e^{(n)}} \varphi(c^{1/p^e}) = R
\]

where \( c \in R \) is some element such that \( R_c \) is regular.

20. So if \( R \) is \( n \)-DFR, then \( \varnothing \) for all \( p \) of height \( h \).

| The question becomes: | Which rings are DFR? |

21. Facts about Diagonal \( F \)-regularity: regular rings are DFR (exercise! Follows from Kunz’s theorem), Segre products of polynomial rings are DFR [CS18] (“non-effective” USTP was known prior to this), tensor products of DFR \( k \)-algebras are DFR [CS18] (new rings with USTP!). DFR rings are strongly \( F \)-regular. DFR rings are not always Gorenstein and can have arbitrarily small \( F \)-signature.

22. **Exercise (hard):** if \( p \) is a height 1 prime and torsion element of the divisor class group, then \( p^{(n)} \neq p^n \) for \( n \gg 0 \). So DFR rings have torsion free divisor class groups [CS18].

### 4 Diagonal \( F \)-regularity of Hibi rings

23. A Hibi ring is a kind of (toric) ring associated to a finite partially ordered set.

24. **Definition** Let \( P = \{v_1, \ldots, v_n\} \) be a poset. The associated Hibi ring, \( k[P] \subseteq k[x_0, \ldots, x_n] \) is defined as follows: we let \( \overline{P} = P \cup \{v_0\} \) where \( v_0 \leq v_i \) for all \( i \). Then:

\[
k[P] := k \left[ \frac{x_0^{a_0} \cdots x_n^{a_n}}{x_I x_J - x_{I \cup J} x_{I \cap J}} \right]
\]

25. If you know about poset ideals, then we can also write

\[
k[P] = k \left[ x_I \mid I \subseteq \overline{P} \text{ a poset ideal} \right] / x_I x_J - x_{I \cup J} x_{I \cap J}
\]
26. We usually denote posets by Hasse diagrams: nodes represent elements of $\overline{P}$. Bigger elements are written above smaller elements. Draw an edge between two distinct nodes $v_i$ and $v_j$ if there's no node between them, i.e. if $v_i \leq v_k \leq v_j$ implies $v_k = v_i$ or $v_k = v_j$. In this case, we say $v_j$ covers $v_i$.

27. Some examples/exercises:

$$\overline{P} = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array}$$

$k[P] =$?

$$\overline{Q} = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array}$$

$k[Q] =$?

28. Checking whether a Hibi ring is $n$-DFR boils down to solving a complicated combinatorial problem:

**Theorem 3 ([PST18]).** For each $i$, let $r_i$ be the length of the longest chain going up from $v_i$ in $P$. Then $k[P]$ is $n$-DFR if and only if there exists some $e$ such that the following holds: for $0 \leq i \leq d$ and $1 \leq m \leq n$, let $\alpha_{i,m}$ be integers in $[0, p^e - 1]$ such that $\sum_{m=1}^{n} \alpha_{j,m} \equiv r_j \pmod{p^e}$ for all $j$. Set $N_j = \left[\sum_{m=1}^{n} \frac{\alpha_{j,m}}{p^e}\right]$. For all $i, j,$ and $m$, let $\varepsilon_{j,i,m} = 1$ if $\alpha_{j,m} > \alpha_{i,m}$ and let $\varepsilon_{j,i,m} = 0$ otherwise. Then there exist $\delta_{i,m} \in \mathbb{Z}$ with

(a) $\delta_{i,m} \geq 0$ for all $m$ whenever $v_i$ is maximal in $P$,

(b) $\delta_{j,m} \leq \varepsilon_{j,i,m} + \delta_{i,m}$ for all $m$ whenever $v_j$ covers $v_i$, and

(c) $\sum_{m=1}^{n} \delta_{j,m} = N_j$

29. **Aside for experts:** The point is that solving this combinatorial problem is the same as constructing a lifting $(R \otimes \mathbb{F})^{1/p^e} \to R^{\otimes n}$ of a map $R^{1/p^e} \to R$ that sends $z = x_0^{r_0} \cdots x_n^{r_n}$ to 1. Note that $z \in R$ and $R_z$ is regular.

30. Using this combinatorial description, we were able to show:

**Theorem 4 ([PST18]).** If $k[P]$ is DFR, so is $k[P \cup \{v'\}]$, where $v'$ covers a single element of $P$. 
31. Example: Checking if $\mathbb{F}_5[x, y, z]$ is 3-DFR:

\[
\begin{array}{ccc}
\alpha_{i,m} & N_i & \delta_{i,m} \\
4 & 4 & 3 \\
4 & 2 & 1 \\
0 & 2 & 1 \\
\end{array}
\]

32. Recall: polynomial rings are DFR. Using theorem 4, which posets (Hasse diagrams) do we know to correspond to DFR Hibi rings?

33. Recall: tensor products of DFR rings are DFR. Here’s what the tensor product of two Hibi rings looks like:

34. **Exercise:** Convince yourself you get isomorphic rings doing the tensor product in either order!

35. **Exercise:** What are all the Hibi rings known to be DFR, using Theorem 4 and results about DFR rings in item 20?

36. **Definition:** A *top node* in a poset is a node that covers more than one element. They look like hats in the Hasse diagram.

**Theorem 5** ([PST18]). *The Hibi ring $k[P]$ is DFR whenever the set of top nodes of $P$ is*

37. The converse to this theorem is not known! Here’s the first poset with incomparable top nodes:

38. We know it’s 2-DFR (in fact, all Hibi rings are 2-DFR). Dylan Johnson has shown it’s 3-DFR.
5 Questions I would like to know the answer to

39. Is the diagonal $F$-regularity of a toric ring independent of characteristic?

40. Is $\mathcal{D}^{(n)}(R)$ a good metric for the singularities of $R$? For instance, if $\mathcal{D}^{(2)}_e(R) = \text{Hom}_R(R^{1/p^e}, R)$ for all $e$, does that imply $R$ is regular? This is true for toric $\mathbb{Q}$-Gorenstein $R$.

41. Do we always have $\mathcal{D}^{(n)}_e(R) \subseteq \mathcal{D}^{(n+1)}_e(R)$? This is true for toric $R$.

42. Are rings with large $F$-signature (say, $>1/2$) always DFR? Note that such rings have torsion-free divisor class groups by Carvajal-Rojas.

43. What kind of USTP statements can we get if the $F$-signature of $\mathcal{D}^{(n)}$ is large but $<1$?

References


