Homework 8.1 Solutions
Math 5110/6830

1. (a) Current:

\[ I_X = \bar{g}_X \omega (V - V_X) \]

(b) It will be hyperpolarized (more negative) since ions are flowing into the cell.

(c) \( V_X \) is \(-100\text{mV} < -65\text{mV}\).

(d) Current-balance equation:

\[ \frac{C}{dt} = -\bar{g}_X \omega (V - V_X) + I_{app} \]

(e) Equilibria point:

\[ V^* = \frac{I_{app}}{\bar{g}_X} + V_X \]

For stability, let

\[ f(V) = \frac{1}{C} [ -\bar{g}_X \omega (V - V_X) + I_{app} ] \]

\[ f'(V) = -\frac{\bar{g}_X \omega}{C} \]

So, \( V^* \) is stable.

(f) No, the parameters are all positive & the derivative is always negative.

(g) Equation:

\[ \frac{C}{dt} = -[\bar{g}_X \omega X (V - V_X) + \bar{g}_Y \omega Y (V - V_Y)] + I_{app} \]

Steady state voltage:

\[ V^* = \frac{\bar{g}_X \omega X V_X + \bar{g}_Y \omega Y V_Y}{\bar{g}_X \omega X + \bar{g}_Y \omega Y} \]
1. Recall that \( I_{ss}(V^*) = I_{ion}(V^*, \omega^*) \)

\[
J(V^*, \omega^*) = \begin{bmatrix}
-\frac{1}{r_\omega} (\bar{g}_{ca} m_\infty + \bar{g}_k \omega^* + \bar{g}_L) & \frac{1}{r_\omega} (\bar{g}_k V^* - \bar{g}_k V_k) \\
-\phi \omega & -\phi \frac{1}{r_\omega}
\end{bmatrix}
\]

\[
det(J) = \phi \frac{1}{r_\omega} [\bar{g}_{ca} m_\infty + \bar{g}_k \omega_\infty + \bar{g}_L + \bar{g}_K \omega_\infty (V^* - V_K)]
\]

If \( det(J) = 0 \), then \( \bar{g}_{ca} m_\infty + \bar{g}_k \omega_\infty + \bar{g}_L + \bar{g}_K \omega_\infty (V^* - V_K) = 0 \). Note that \( \frac{dI}{dV} = \bar{g}_{ca} m_\infty + \bar{g}_k \omega_\infty + \bar{g}_L \). The only way to satisfy this is to have \( V^* = V_K \) and \( \frac{dI}{dV} = 0 \). In addition, if \( I_{ss} \) is monotone, then \( |\frac{dI}{dV}| > 0 \) for all \( V \) and \( det(J) \neq 0 \).

2. (a) Plot of \( I_{ss}(V) \):

(b) See graph in part (c).

(c) Nullclines with \( I_{app} = 0 \):
(d) If $I_{app}$ is held at $-50$ for sufficiently long, then the solution will approach the new steady state (shown in the figure below); however, once $I_{app}$ is turned off then it returns back to the original steady state.

3. (a) Steady states for different values of $I_{app}$:
(b) There is clearly a bifurcation happening since we lose/gain equilibria points. As you can see in the following figure, we go from having one equilibria point ($I_{app} = 50$), to having two (not shown here), to having three ($I_{app} = 0$), to again having two (the not shown), and finally one ($I_{app} = -50$).