1. (a) For

\[ \frac{dx}{dt} = x - x^3 \]

Phase-line diagram:

Fixed points are \( x^* = 0 \) and \( x^* = \pm 1 \). Stability analytically (you can get this from your phase-line diagram too). Let

\[
\begin{align*}
  f(x) &= x - x^3 \\
  f'(x) &= 1 - 3x^2 \\
  f'(0) &= 1 \\
  f'(1) &= -2 \\
  f'(-1) &= -2
\end{align*}
\]

Then, \( x^* = 0 \) is unstable and both \( x = 1 \) and \( x = -1 \) are stable.

Sketch of \( x(t) \):
To solve $\dot{x} = x - x^3$, we can use separation of variables.

\[
\frac{1}{x - x^3} \, dx = dt
\]

\[
\int_{x_0}^{x(t)} \frac{1}{x - x^3} \, d\bar{x} = \int_0^t \, d\bar{t}
\]

\[
\int_{x_0}^{x(t)} \frac{1}{\bar{x}(1 - \bar{x})(1 + \bar{x})} \, d\bar{x} = t
\]

Using partial fractions:

\[
\int_{x_0}^{x(t)} \left( \frac{1}{\bar{x}} + \frac{\frac{1}{2}}{1 - \bar{x}} - \frac{\frac{1}{2}}{1 + \bar{x}} \right) \, d\bar{x} = t
\]

\[
\left[ \ln(\bar{x}) - \frac{1}{2} \ln(1 - \bar{x}) - \frac{1}{2} \ln(1 + \bar{x}) \right]_{x_0}^{x(t)} = t
\]

\[
\left[ \ln \left( \frac{\bar{x}}{\sqrt{1 - \bar{x}^2}} \right) \right]_{x_0}^{x(t)} = t
\]

\[
\ln \left( \frac{x}{\sqrt{1 - x^2}} \right) - \ln \left( \frac{x_0}{\sqrt{1 - x_0^2}} \right) = t
\]

\[
\ln \left( \frac{x^2}{1 - x^2} \right)^{1/2} = t + \ln \left( \frac{x_0^2}{1 - x_0^2} \right)^{1/2}
\]

\[
\frac{1}{2} \ln \left( \frac{x^2}{1 - x^2} \right) = t + \frac{1}{2} \ln \left( \frac{x_0^2}{1 - x_0^2} \right)
\]

\[
\ln \left( \frac{x^2}{1 - x^2} \right) = 2t + \ln \left( \frac{x_0^2}{1 - x_0^2} \right)
\]

\[
x^2 = \frac{x_0^2}{1 - x_0^2} e^{2t}
\]

\[
x^2 (1 + \frac{x_0^2}{1 - x_0^2} e^{2t}) = \frac{x_0^2}{1 - x_0^2} e^{2t}
\]

\[
x^2 = \frac{x_0^2}{1 - x_0^2} e^{2t}
\]

\[
x(t) = \sqrt{\frac{x_0^2}{1 - x_0^2} e^{2t}}
\]

\[
x(t) = \sqrt{\frac{x_0^2 e^{2t}}{(1 - x_0^2 + x_0^2 e^{2t})}}
\]

\[
x(t) = \frac{x_0 e^t}{\sqrt{(1 - x_0 + x_0^2 e^{2t})}}
\]
(b) For
\[ \frac{dx}{dt} = e^{-x} \sin(x) \]

Phase-line diagram:

Fixed points are \( x^* = n\pi \). Stability analytically (you can get this from your phase-line diagram too).
Let
\[ f(x) = e^{-x} \sin(x) \]
\[ f'(x) = -e^{-x} \sin(x) + e^{-x} \cos(x) \]

Then, \( x^* = n\pi \) is unstable for \( n \) even and stable for \( n \) odd.

Sketch of \( x(t) \):

This cannot be solve analytically.

2. To find an equation for this phase-line diagram, we can notice that we will have roots \( x = -1 \), \( x = 0 \) and \( x = 2 \). So, the equation will take the form
\[ \frac{dx}{dt} = x(x + 1)(x - 2) \]

But, to get the correct stability, we can check concavity. We need the function to be concave up at \( x = -1 \), concave down at \( x = 0 \) and concave up at \( x = 2 \):
\[ f(x) = x(x + 1)(x - 2) \]
\[ f'(x) = 3x^2 - 2x - 2 \]
\[ f''(x) = 6x - 2 \]
\[ f''(-1) = -8 \]
So, this fails. Let’s think about it though. If we want this equation to be concave up at $x = -1$ then the term $(x + 1)$ should be squared. Let’s try $\frac{dx}{dt} = x(x + 1)^2(x - 2)$:

$$f''(x) = 12x^2 - 6$$
$$f''(-1) = 6$$
$$f''(0) = -6$$
$$f''(2) = 42$$

Then, this works! So, an equation to match the phase-line diagram is:

$$\frac{dx}{dt} = x(x + 1)^2(x - 2)$$

3. We can use separation of variables for this equation:

$$\frac{dN}{dt} = K(t)N$$
$$\frac{1}{N}dN = K(t)dt$$

$$\int_{N_0}^{N(t)} \frac{1}{N}d\tilde{N} = \int_0^t K(\tilde{t})d\tilde{t}$$

$$\ln(\tilde{N})\bigg|_{N_0}^{N(t)} = \int_0^t K(\tilde{t})d\tilde{t}$$

$$\ln(N(t)) - \ln(N_0) = \int_0^t K(t)d\tilde{t}$$

$$\ln\left(\frac{N(t)}{N_0}\right) = \int_0^t K(t)d\tilde{t}$$

$$\frac{N(t)}{N_0} = e^{\int_0^t K(t)d\tilde{t}}$$

$$N(t) = N_0e^{\int_0^t K(t)d\tilde{t}}$$

4. Solution using separation of variables:

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)$$

$$\frac{1}{N \left(1 - \frac{N}{K}\right)}dN = rdt$$

$$\int_{N_0}^{N(t)} \frac{1}{N \left(1 - \frac{N}{K}\right)}d\tilde{N} = \int_0^t r\tilde{t}$$

Using partial fractions:

$$\int_{N_0}^{N(t)} \left[ \frac{1}{N} + \frac{\frac{1}{K}}{1 - \frac{N}{K}} \right] d\tilde{N} = \int_0^t r\tilde{t}$$
Now integrate:

\[
\left[ \ln \bar{N} - \ln \left( 1 - \frac{\bar{N}}{K} \right) \right]_{N_0}^{N(t)} = rt
\]

\[
\ln \left( \frac{\bar{N}}{1 - \frac{\bar{N}}{K}} \right)_{N_0}^{N(t)} = rt
\]

\[
\ln \left( \frac{N(t)}{1 - \frac{N(t)}{K}} \right) - \ln \left( \frac{N_0}{1 - \frac{N_0}{K}} \right) = rt
\]

\[
\ln \left( \frac{1 - \frac{N_0}{K}}{1 - \frac{N(t)}{K}} \right)_{N_0}^{N(t)} = rt
\]

\[
\frac{K - N_0}{K - N(t)} N(t) = N_0 e^{rt}
\]

\[(K - N_0)N(t) = N_0 e^{rt} (K - N(t))\]

\[(K - N_0 + N_0 e^{rt})N(t) = N_0 Ke^{rt}\]

\[
N(t) = \frac{N_0 Ke^{rt}}{K - N_0 + N_0 e^{rt}}
\]

\[
N(t) = \frac{N_0 e^{rt}}{1 + \frac{N_0}{K} (e^{rt} - 1)}
\]

**Solution using the substitution** \( x = \frac{1}{N} \):

With \( x = \frac{1}{N} \), or equivalently \( N = \frac{1}{x} \), we have:

\[
\frac{dN}{dt} = -\frac{1}{x^2} \frac{dx}{dt}
\]

Substituting this into the logistic eqn:

\[
-\frac{1}{x^2} \frac{dx}{dt} = r \frac{1}{x} \left( 1 - \frac{1}{K} \frac{1}{x} \right)
\]

\[
\frac{dx}{dt} = r \left( \frac{1}{K} - x \right)
\]

Now separate variables and integrate:

\[
\int_{x_0}^{x(t)} \frac{1}{K - \bar{x}} \, d\bar{x} = \int_0^t \frac{rd\bar{t}}{1 + \frac{N_0}{K} (e^{rt} - 1)}
\]

\[
-\ln \left( \frac{1}{K} - \bar{x} \right)_{x_0}^{x(t)} = rt
\]

\[-\ln \left( \frac{1}{K} - x(t) \right) + \ln \left( \frac{1}{K} - x_0 \right) = rt\]

\[
\ln \left( \frac{1}{K} - x_0 \right) = rt
\]

\[
\frac{1}{K - x_0} = e^{rt}
\]

\[
1 - Kx(t) = (1 - Kx_0) e^{-rt}
\]

\[
x(t) = \frac{1}{K} (1 - (1 - Kx_0) e^{-rt})
\]
Returning to the original variables:

\[
\frac{1}{N(t)} = \frac{1}{K} \left[ 1 - \left( 1 - \frac{K}{N_0} \right) e^{-rt} \right]
\]
\[
N(t) = \frac{K}{1 - \left( 1 - \frac{K}{N_0} \right) e^{-rt}}
\]
\[
= \frac{Ke^{rt}}{e^{rt} - 1 + \frac{K}{N_0}}
\]
\[
= \frac{N_0 e^{rt}}{\frac{N_0}{K} e^{rt} - \frac{N_0}{K} + 1}
\]
\[
= \frac{N_0 e^{rt}}{1 + \frac{N_0}{K} (e^{rt} - 1)}
\]

5. Phase-line diagram:

Solution:
Homework 5.2 Solutions
Math 5110/6830

1. (a) Fixed points satisfy:

\[ \frac{dx}{dt} = 0 = 1 - e^{-x^2} \]

The only fixed point is \( x^* = 0 \). To determine its stability:

\[ f(x) = 1 - e^{-x^2} \]
\[ f'(x) = 2xe^{-x^2} \]
\[ f'(0) = 0 \]

From this, we cannot determine the stability of \( x^* = 0 \). Therefore, we will need to do this graphically: As you can see, this point is half-stable.

(b) Fixed points satisfy:

\[ \frac{dx}{dt} = 0 = \ln(x) \]

The only fixed point is \( x^* = 1 \). To determine its stability:

\[ f(x) = \ln(x) \]
\[ f'(x) = \frac{1}{x} \]
\[ f'(1) = 1 \]

And, \( x^* = 1 \) is unstable.

(c) Fixed points satisfy:

\[ \frac{dx}{dt} = 0 = x(1-x)(2-x) \]

The fixed points are \( x^* = 0, x^* = 1, \) and \( x^* = 2 \). To determine their stability:

\[ f(x) = x(1-x)(2-x) \]
\[ f'(x) = (1-x)(2-x) - x(2-x) - x(1-x) \]
\[ f'(0) = 2 \]
\[ f'(1) = -1 \]
\[ f'(2) = 2 \]

And, \( x^* = 0 \) is unstable, \( x^* = 1 \) is stable and \( x^* = 2 \) is unstable.
(d) Fixed points satisfy:
\[
\frac{dx}{dt} = 0 = ax - x^3 = x(a - x^2)
\]
The fixed points are \(x^* = 0\), \(x^* = -\sqrt{a}\), and \(x^* = \sqrt{a}\). To determine their stability, let
\[f(x) = ax - x^3\]
Then, for \(a > 0\)
\[
\begin{align*}
f'(x) &= a - 3x^2 \\
f'(0) &= a \\
f'(-\sqrt{a}) &= -2a \\
f'((\sqrt{a}) &= -2a
\end{align*}
\]
For \(a > 0\), \(x^* = 0\) is unstable, \(x^* = -\sqrt{a}\) is stable and \(x^* = \sqrt{a}\) is stable. But for \(a < 0\), \(x^* = 0\) is stable, \(x^* = -\sqrt{a}\) is unstable and \(x^* = \sqrt{a}\) is unstable. And, for \(a = 0\), there is only one fixed point \(x^* = 0\) which we need to do a graphical stability analysis for:

Looks stable to me.

2. (a) To determine stability, let
\[
\begin{align*}
f(x) &= -x^c \\
\text{and} \\
f'(x) &= -cx^{c-1}
\end{align*}
\]
Then, for \(c = 1\) this is stable. However, we need to do this graphically for other values of \(c\): ????

(b) To find an equation for time here, we can separate variables and solve this DE:
\[
\begin{align*}
\frac{dx}{dt} &= -x^c \\
\int_{x_0}^{x(t)} \frac{1}{x^c} \, dx &= \int_0^t -d\bar{t} \\
\int_{x_0}^{x(t)} \bar{x}^{-c} d\bar{x} &= \int_0^t -d\bar{t} \\
\left( \bar{x}^{-c+1} \right)_{x_0}^{x(t)} &= -t \\
x(t)^{-c+1} - x_0^{-c+1} &= -t \\
x(t)^{-c+1} &= x_0^{-c+1} - t(-c + 1) \\
x(t) &= \left[ x_0^{-c+1} - t(-c + 1) \right]^{-\frac{1}{c-1}}
\end{align*}
\]
Letting $x_0 = 1$:

$$x(t) = [1 - t(-c + 1)]^{-\frac{2}{c+1}}$$

Now, to find the time it takes til $x(t) = 0$:

$$x(t) = 0 = [1 - t(-c + 1)]^{-\frac{2}{c+1}}$$

$$0 = 1 - t(-c + 1)$$

$$t = \frac{1}{1-c}$$

3. To show that there are infinite solutions to $\dot{x} = x^{1/3}$, we can find at least 2 solutions that pass through the same point. Notice that $x(t) = 0$ is one solution that passes through the point $(0, 0)$. The other solution, found by separation of variables, is $x(t) = \frac{2}{3} x^{3/2}$. This solution also passes through the point $(0, 0)$. 