Homework 2.1 Solutions
Math 5110/6830

1. (a) The variables, parameters & terms:

\[ a_n = \text{whale population after } n \text{ years} \]
\[ k = \text{growth rate} \]
\[ M = \text{carrying capacity} \]
\[ m = \text{minimum survival level} \]

- If \( 0 < a_n < m \), then \( k(M - a_n)(a_n - m) < 0 \) and the population will decline.
- If \( m < a_n < M \), then \( k(M - a_n)(a_n - m) > 0 \) and the population will grow.
- If \( a_n > M \), then \( k(M - a_n)(a_n - m) < 0 \) and the population will decline.
- If \( a_n = m \) or \( a_n = M \), then \( k(M - a_n)(a_n - m) = 0 \) and the population will remain the same (ie. \( a_n = m \) and \( a_n = M \) are fixed points).

(b) To find fixed points, we set \( a_{n+1} = a_n = a^* \):

\[ a^* = a^* + k(M - a^*)(a^* - m) \]
\[ 0 = k(M - a^*)(a^* - m) \]

\[ a^* = M \quad \text{OR} \quad a^* = m \]

With \( M = 5000, \ m = 100, \) and \( k = 0.0001 \), we have fixed points \( a^* = 5000 \) and \( a^* = 100 \). To find the stability, let

\[ f(a) = a + 0.0001(5000 - a)(a - 100) \]
\[ = a + 0.0001(5100a - 500000 - a^2) \]

Then,

\[ f'(a) = 1 + 0.0001(5100 - 2a) \]

For \( a^* = 100 \):

\[ |f'(100)| = |1 + 0.0001(5100 - 200)| \]
\[ = 1.49 > 1 \]

So, \( a^* = 100 \) is unstable.

For \( a^* = 5000 \):

\[ |f'(100)| = |1 + 0.0001(5100 - 10000)| \]
\[ = 0.51 < 1 \]

So, \( a^* = 5000 \) is stable.
(c) The following cobweb graphically says what we just found: \( a^* = 100 \) is unstable and \( a^* = 5000 \) is stable.

(d) The following represents solutions for different initial conditions:

(e) When \( a_0 < m \), the population declines and eventually becomes negative. When \( a_0 >> M \), then \( a_1 < 0 \) and the population continues to decline. Both of these cases are problematic since we can’t have a negative number of whales. I don’t think the whales would appreciate that!!

2. (a) The system we have is:

\[
M_{t+1} = M_t - \frac{M_p^t}{K^p + M_p^t} M_t + S
\]

With \( K = 2 \) and \( S = 1 \):

\[
M_{t+1} = M_t - \frac{M_p^t}{2^p + M_p^t} M_t + 1
\]
A fixed point would satisfy:

\[ M^* = M^* - \frac{M^{*p}}{2p + M^{*p}} M^* + 1 \]

With \( M^* = 2 \):

\[
2 - \frac{2^p}{2^p + 2^p} 2 + 1 = 3 - \frac{2^p}{2(2^p)^2} \\
= 3 - 1 \\
= 2
\]

This is true for all values of \( p \).

(b) To find the stability of this system for the fixed point \( M^* = 2 \), we need to first calculate a derivative. Let

\[ g(M) = M - \frac{M^p}{2^p + M^p} M + 1 \]

Then,

\[ g'(M) = 1 - \frac{pM^{p-1}(2^p + M^p) - M^p p M^{p-1}}{(2^p + M^p)^2} M - \frac{M^p}{2^p + M^p} \]

With \( M^* = 2 \):

\[
g'(2) = 1 - \frac{p 2^{p-1}(2^p)}{(2^p + 2^p)^2} 2 - \frac{2^p}{2^p + 2^p} \\
= 1 - \frac{p 2^{2p}}{4(2^{2p})} - \frac{2^p}{2(2^p)} \\
= \frac{1}{2} - \frac{p}{4}
\]

For \( M^* = 2 \) to be stable, we need \( |g'(2)| < 1 \). So, this happens when \(-1 < g'(2) < 1\):

\[-1 < \frac{1}{2} - \frac{p}{4} < 1 \\
-\frac{3}{2} < -\frac{p}{4} < \frac{1}{2} \\
\frac{3}{2} > \frac{p}{4} > -\frac{1}{2} \\
6 > p > -2
\]

Then, \( M^* = 2 \) is stable if \(-2 < p < 6 \) and unstable if \( p < -2 \) or \( p > 6 \). The solution will oscillate when \( g'(2) < 0 \), i.e. when \( p > 2 \).
(c) Cobweb for when $M^* = 2$ is unstable:
1. (a) 

\[ f^2(x) = r(rx(1-x))(1-(rx(1-x))) \]
\[ = r^2x(1-(r+1)x + 2rx^2 - rx^3) \]

(b) First, we know that fixed points of \( f(x) \) will also be fixed points of \( f^2(x) \). So, from \( f(x) \):

\[ x^* = r(1-x)x \]
\[ 0 = (r-rx-1)x \]

Then, fixed points of both \( f \) and \( f^2 \) are \( x^* = 0 \) and \( x^* = \frac{r-1}{r} \). However, there are other fixed points of \( f^2 \). To find these, first write:

\[ x^* = r^2x^*(1-(r+1)x^* + 2rx^2 - r(x^*)^3) \]
\[ 0 = (r^2(1-(r+1)x^* + 2rx^2 - r(x^*)^3) - 1)x^* \]

We can already see that \( x^* = 0 \) is a solution to this. Then,

\[ 0 = r^2(1-(r+1)x^* + 2rx^2 - r(x^*)^3) - 1 \]
\[ \frac{1}{r^2} = 1 - (r+1)x^* + 2rx^2 - r(x^*)^3 \]
\[ 0 = \left(1 - \frac{1}{r^2}\right) - (r+1)x^* + 2rx^2 - r(x^*)^3 \]
\[ 0 = \frac{r^2 - 1}{r^3} - \left(1 + \frac{1}{r}\right)x^* + 2(x^*)^2 - (x^*)^3 \]

And, since we also know that \( x^* = \frac{r-1}{r} \) is a fixed point:

\[ 0 = \left(x^* - \frac{r-1}{r}\right)\left((x^*)^2 - \left(\frac{r+1}{r}\right)x^* + \left(\frac{r+1}{r^2}\right)\right) \]

Then, with a little algebra

\[ x^* = 0 \]
\[ x^* = \frac{r-1}{r} \]
\[ x^* = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r} \]

The first two are trivial 2-cycles, so let’s take a look at the third fixed point. If \( r < 3 \), the roots are imaginary so there is no 2-cycle. If \( r = 3 \) then, there is only one root so there is no 2-cycle. But, if \( r > 3 \), then we get two distinct roots & a nontrivial 2-cycle.

(c) Computing \( \frac{d}{dx} f^2(x) \):

\[ \frac{d}{dx} f^2(x) = -4r^3x^3 + 6r^3x^2 - 2(r^2 + r^3)x + r^2 \]

(d) To do this, we need to evaluate \( |\frac{d}{dx} f^2(x^*)| \):

\[ |\frac{d}{dx} f^2(x^*)| = | -r^2 + 2r + 4 | \]

Then, \( |\frac{d}{dx} f^2(x^*)| < 1 \) (ie. stable) when \(-1 < -r^2 + 2r + 4 < 1 \). Solving each of these inequalities should yield that for the 2-cycle to be stable we need \( r > 3 \) and \( r < 1 + \sqrt{6} \). It is unstable elsewhere.
2. (a) Graph of $g(x_n)$ versus $x_n$:

(b) Fixed points satisfy:

\[
x^* = \frac{r}{1 + \frac{r-1}{K} x^*}
\]

\[
1 = \frac{r}{1 + \frac{r-1}{K} x^*}
\]

\[
1 + \frac{r-1}{K} x^* = r
\]

\[
\frac{r-1}{K} x^* = r - 1
\]

\[
x^* = 0 \text{ OR } x^* = K
\]

(c) Let

\[
f(x) = \frac{r}{1 + \frac{r-1}{K} x}
\]

Then,

\[
f'(x) = \frac{r}{(1 + \frac{r-1}{K} x)^2}
\]

For $x^* = 0$:

\[
|f'(x)| = |r|
\]

This is \textit{stable} for $0 < r < 1$ and \textit{unstable} for $r > 1$.

For $x^* = K$:

\[
|f'(x)| = \left| \frac{1}{r} \right|
\]

This is \textit{stable} for $r > 1$ and \textit{unstable} for $0 < r < 1$. 
Bifurcation Diagram:

(d) Cobweb for $r > 1$ (left) and $r < 1$ (right):

(e) Solutions:

(f) So far this system doesn’t look like it’ll have cycles or chaos. We can tell this from the behavior of the cobweb plot, solutions, and bifurcation diagram.

(g) We have

$$x_{n+1} = \frac{r}{1 + \frac{r-1}{K}x_n} x_n$$

Let $u_n = \frac{1}{x_n}$. Then,

$$\frac{1}{u_{n+1}} = \frac{r}{1 + \frac{r-1}{K}x_n} \frac{1}{x_n}$$

$$u_{n+1} = \frac{1}{r} \left( 1 + \frac{r-1}{K} u_n \right) u_n$$

$$= \frac{1}{r} u_n + \frac{r - 1}{rK} u_n$$
Rewrite this as \( u_{n+1} = Au_n + B \), where \( A = \frac{1}{r} \) and \( B = \frac{r-1}{rK} \). Then, we can find a solution by doing the following:

\[
\begin{align*}
  u_{n+1} &= Au_n + B \\
         &= A(Au_{n-1} + B) + B = A^2u_{n-1} + B(A + 1) \\
         &= A^2(Au_{n-2} + B) + B(A + 1) = A^3u_{n-1} + B(A^2 + A + 1) \\
         &\vdots \\
         &= A^{n+1}u_0 + B(A^n + A^{n-1} + \ldots + A + 1) \\
         &= A^{n+1}u_0 + B \frac{A^{n+1} - 1}{A - 1}
\end{align*}
\]

Now, return back to the original variables:

\[
\begin{align*}
  \frac{1}{x_{n+1}} &= \frac{1}{r^{n+1}x_0} + \frac{r - 1}{rK} \frac{1}{r^{n-1} - 1} \\
  x_{n+1} &= \frac{r^{n+1}x_0}{1 + \frac{r^{n+1} - 1}{K}x_0}
\end{align*}
\]

3. (a) For this system we have:

\[
x_{n+1} = exp \left[ r \left( 1 - \frac{x_n}{K} \right) \right] x_n
\]

Plot of \( g(x_n) \) for \( r=1, K=10 \):

![Plot of g(x_n) for r=1, K=10](image)

(b) Fixed points satisfy:

\[
\begin{align*}
  x^* &= exp \left[ r \left( 1 - \frac{x^*}{K} \right) \right] x^* \\
  1 &= exp \left[ r \left( 1 - \frac{x^*}{K} \right) \right] \\
  0 &= r \left( 1 - \frac{x^*}{K} \right)
\end{align*}
\]

\[
x^* = 0 \quad \text{OR} \quad x^* = K
\]

(c) Let

\[
f(x) = exp \left[ r \left( 1 - \frac{x}{K} \right) \right] x
\]
Then,

\[ f'(x) = \left(1 - \frac{rx}{K}\right) \exp \left[ r \left(1 - \frac{x}{K}\right) \right] \]

For \( x^* = 0 \):

\[ |f'(0)| = |e^r| \]

This is always unstable.

For \( x^* = K \):

\[ |f'(0)| = |1 - r| \]

This is stable if \( 0 < r < 2 \) and unstable for \( r > 2 \).

Bifurcation Diagram:

(d) Cobweb:

Solutions: