1. (a) The fixed points satisfy:

\[ V^* = -w^*(V^* - V_0) + V^* \]
\[ w^* = \alpha f(V^*) + (1 - \alpha)w^* \]
\[ = \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} + (1 - \alpha)w^* \]

From the first equation:
\[ V^* = V_0 \]

Then, from the second eqn:
\[ w^* = \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} + (1 - \alpha)w^* \]
\[ w^* - (1 - \alpha)w^* = \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} \]
\[ \alpha w^* = \frac{\alpha}{1 + e^{-\beta(V^* - V_0)}} \]
\[ w^* = \frac{1}{1 + e^{-\beta(V^* - V_0)}} \]

Plugging \( V^* = V_0 \) into the above eqn:
\[ w^* = \frac{1}{2} \]

So, the fixed points are:
\[ V^* = V_0 \]
\[ w^* = \frac{1}{2} \]

(b) To study the fixed points of this system, we need to find a Jacobian:

\[ J(V^*, w^*) = \begin{pmatrix} -w^* + 1 & -V^* + V_0 \\ \alpha f'(V^*) & 1 - \alpha \end{pmatrix} \]

Then,
\[ J\left(V_0, \frac{1}{2}\right) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 - \alpha \end{pmatrix} \]

To check the stability of this point, you can either use the Jury conditions (\(|Tr(J)| < 1 + Det(J) < 2\)) or look at the evals. With \( \alpha = 0.2 \):

\[ Tr(J) = 1.3 \]
\[ Det(J) = 0.4 \]

Since \( 1.3 < 1.4 < 2 \), then, \( (V_0, \frac{1}{2}) \) is stable. Looking at the evals, we find that \( \lambda_1 = 0.5 \) and \( \lambda_2 = 0.8 \). Since these are less than 1, then this point is stable.
(c) Phase-plane solutions (for \( V_0 = 1 \)):

\[ K^{*} = K^{*} + p(N - K^{*})K^{*} \]
\[ = K^{*}(1 + p(N - K^{*})) \]

Then,
\[ K^{*} = 0 \]
\[ K^{*} = N \]

(c) To check the stability of these points, we need to take a derivative and find when \( |f'(K^{*})| < 1 \). Let
\[ f(K) = K + p(N - K)K \]

Then,
\[ f'(K) = 1 + p(N - K) - pK \]
\[ = 1 + p(N - 2K) \]

And,
\[ f'(0) = 1 + pN \]
\[ f'(N) = 1 - pN \]

For \( p = 0.5 \):
\[ f'(0) = 1 + \frac{N}{2} \]
\[ f'(N) = 1 - \frac{N}{2} \]

Then, \( K^{*} = 0 \) is always unstable. And, \( K^{*} = N \) is stable for \( 0 < N < 4 \) and unstable for \( N > 4 \).
(d) Bifurcation Diagram:

(e) Cobwebbing:

(f) Solution:
(g) When $N > 4$, cycles will start to appear. Remember that we cannot have 2 unstable fixed points next to each other (you can see this from the bifurcation diagram). Therefore, we will start to see period doubling. We may see cycles of longer period than 2 as well as chaos.

(h) When $K_0$ is large, the model fails. It predicts that $K_1 < 0$ which is not biologically possible.

3. (a)

$$
R_{n+1} = (1 - \sigma)R_n + \frac{r}{2}G_n \\
G_{n+1} = \left(1 - \sigma + \frac{g}{2}\right)G_n
$$

(b) To determine the conditions for the population of gray birds to grow, we need to find the evals. For the population to grow, the magnitude of the leading eval needs to be $> 1$:

$$
\begin{pmatrix}
R_{n+1} \\
G_{n+1}
\end{pmatrix} = \begin{pmatrix}
1 - \sigma & \frac{r}{2} \\
0 & 1 - \sigma + \frac{g}{2}
\end{pmatrix} \begin{pmatrix}
R_n \\
G_n
\end{pmatrix}
$$

Evals are

$$
\lambda_1 = 1 - \sigma \\
\lambda_2 = 1 - \sigma + \frac{g}{2}
$$

The larger eval is $\lambda_2 = 1 - \sigma + \frac{g}{2}$. We now need to look for when $|\lambda_2| > 1$:

$$
\begin{align*}
1 - \sigma + \frac{g}{2} &> 1 \\
\text{since } 0 < \sigma < 1 \\
1 - \sigma + \frac{g}{2} &> 0
\end{align*}
$$

We can then drop the absolute values:

$$
1 - \sigma + \frac{g}{2} > 1 \\
\text{then} \\
\frac{g}{2} > \sigma
$$

So, for the population to grow, we need $\frac{g}{2} > \sigma$. This means that the number of offspring per bird has to exceed the number of deaths per bird. Biologically, this makes sense!!

(c) If the condition in part (b) is not satisfied, then the population will go extinct.

(d) Let’s start by writing the system in a matrix. Then, we will find the evals, use the ICs to find the constants, & write the solution. Our system (notice this is the same as you already found in part (b)):

$$
\begin{pmatrix}
R_{n+1} \\
G_{n+1}
\end{pmatrix} = \begin{pmatrix}
1 - \sigma & \frac{r}{2} \\
0 & 1 - \sigma + \frac{g}{2}
\end{pmatrix} \begin{pmatrix}
R_n \\
G_n
\end{pmatrix}
$$

We already found the evals in part(b):

$$
\lambda_1 = 1 - \sigma \\
\lambda_2 = 1 - \sigma + \frac{g}{2}
$$

With $\sigma = 0.5$ and $g = 3$:

$$
\lambda_1 = 0.5 \\
\lambda_2 = 2$$
Then, we can write a general solution (with arbitrary constants) as:

\[ R_n = A_1(0.5)^n + A_22^n \]
\[ G_n = B_1(0.5)^n + B_22^n \]

Notice that we have 2 equations but 4 unknowns. The way to take care of this is to rewrite the \( G_n \) equation in terms of the \( R_n \) equation. From our original system, we can rearrange terms to have

\[ G_n = 2R_{n+1} - (1 - \sigma)R_n \]

Plugging the \( R_n \) solution into this:

\[ G_n = \frac{2}{r}(A_1(0.5)^{n+1} + 2A_2^{n+1} - (1 - \sigma)(A_1(0.5)^n + A_22^n)) \]
\[ = \frac{2}{r}(A_1(0.5)^{n+1} + A_22^{n+1} - 0.5(A_1(0.5)^n + A_22^n)) \]
\[ = \frac{2}{r}(0.5A_1(0.5)^n + 2A_22^n - 0.5A_1(0.5)^n - 0.5A_22^n) \]
\[ = 3\frac{r}{r}A_22^n \]

We now have 2 eqns and 2 unknowns:

\[ R_n = A_1(0.5)^n + A_22^n \]
\[ G_n = 3\frac{r}{r}A_22^n \]

Lets use our ICs to find the constants \( A_1 \) and \( A_2 \):

\[ R_0 = 3 = A_1 + A_2 \]
\[ G_0 = 5 = 3\frac{r}{r}A_2 \]

Then,

\[ A_1 = \frac{9 - 5r}{3} \]
\[ A_2 = \frac{5r}{3} \]

And, the solution is:

\[ R_n = \frac{9 - 5r}{3}(0.5)^n + \frac{5r}{3}(2)^n \]
\[ G_n = 5(2)^n \]

(e) No. If \( G_n \) survives, then \( G_n \uparrow \infty \) and \( R_n \to \infty \) (for \( r > 0 \)).