1. Find $\frac{dy}{dx}$ for
   \[(a) \quad (x + y)^3 = x^3 + y^3\]
   
   \[
   3(x + y)^2 (1 + \frac{dy}{dx}) = 3x^2 + 3y^2 \frac{dy}{dx} \\
   3(x + y)^2 + 3(x + y)^2 \frac{dy}{dx} = 3x^2 + 3y^2 \frac{dy}{dx} \\
   3(x + y)^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 3x^2 - 3(x + y)^2 \\
   \frac{dy}{dx} [3(x + y)^2 - 3y^2] = 3x^2 - 3(x + y)^2 \\
   \frac{dy}{dx} = \frac{3x^2 - 3(x + y)^2}{3(x + y)^2 - 3y^2} \\
   \frac{dy}{dx} = -y(2x + y) \\
   \frac{dy}{dx} = \frac{x(x + 2y)}{x(x + 2y)} \\
   
   (b) \quad y^2 = \frac{x^3}{4 - x} \text{ at } (2, 2)
   
   \[
   2y \frac{dy}{dx} = \frac{3x^2(4 - x) - x^3(-1)}{(4 - x)^2} \\
   2y \frac{dy}{dx} = \frac{12x^2 - 3x^3 + x^3}{(4 - x)^2} \\
   \frac{dy}{dx} = \frac{12x^2 - 2x^3}{2y(4 - x)^2} \\
   \frac{dy}{dx} = \frac{x^2(6 - x)}{y(4 - x)^2} \\
   \frac{dy}{dx}(2, 2) = \frac{2^2(6 - 2)}{2(4 - 2)^2} = \frac{16}{8} = 2
   
2. Find the critical values AND inflection points for:
   \[(a) \quad f(x) = (x + 2)^{2/3}\]
   
   Critical Values:
   \[
   f'(x) = \frac{2}{3}(x + 2)^{-1/3} = \frac{2}{3(x + 2)^{1/3}} \\
   f'(x) = 0 \Rightarrow \text{no critical values from this!} \\
   f'(x) = \text{undefined} \Rightarrow x = -2
   
   Inflection Points:
   \[
   f''(x) = \frac{-2}{9(x + 2)^{4/3}} \\
   f''(x) = 0 \Rightarrow \text{no inflection points from this!} \\
   f''(x) = \text{undefined} \Rightarrow x = -2
(b) \( f(x) = \sqrt{x^2 - 1} \)

Critical Values:

\[
f'(x) = \frac{1}{2} (x^2 - 1)^{-1/2} (2x) = \frac{x}{\sqrt{x^2 - 1}}
\]

\[
f'(x) = 0 \Rightarrow x = 0
\]

\[
f'(x) = \text{undefined} \Rightarrow x = -1 \text{ and } x = 1
\]

Inflection Points:

\[
f''(x) = \frac{-1}{(x^2 - 1)^{3/2}}
\]

\[
f''(x) = 0 \Rightarrow \text{no inflection points from this!}
\]

\[
f''(x) = \text{undefined} \Rightarrow x = -1 \text{ and } x = 1
\]

3. For \( f(x) = 6x^3 - 15x^2 + 12x \), find:

(a) critical values

\[
f'(x) = 18x^2 - 30x + 12
\]

\[
f'(x) = 0 \Rightarrow x = 1 \text{ and } x = \frac{2}{3}
\]

\[
f'(x) = \text{undefined} \Rightarrow \text{no critical values from this!}
\]

(b) increasing and decreasing intervals

\[
\begin{array}{l|c|c|c}
\text{Intervals} & (-\infty, \frac{2}{3}) & (\frac{2}{3}, 1) & (1, \infty) \\
\text{Test Value} & x = 0 & x = \frac{2}{3} & x = 2 \\
\text{Sign of f'(x)} & + & - & + \\
\text{Inc/Dec} & \text{INC} & \text{DEC} & \text{INC} \\
\end{array}
\]

(c) inflection points

\[
f''(x) = 36x - 30
\]

\[
f''(x) = 0 \Rightarrow x = \frac{5}{6}
\]

\[
f''(x) = \text{undefined} \Rightarrow \text{no inflection points from this!}
\]

(d) concavity intervals

\[
\begin{array}{l|c|c}
\text{Intervals} & (-\infty, \frac{5}{6}) & (\frac{5}{6}, \infty) \\
\text{Test Value} & x = 0 & x = 1 \\
\text{Sign of f''(x)} & - & + \\
\text{Concave up/down} & \text{DOWN} & \text{UP} \\
\end{array}
\]

(e) all extrema

Relative Max: \( x = \frac{2}{3} \)

Relative Min: \( x = 1 \)
4. For \( f(x) = x^3 - 3x^2 \) on the interval \([-1,3]\), find:

(a) critical values

\[
\begin{align*}
 f'(x) &= 3x^2 - 6x \\
 f'(x) &= 0 \Rightarrow x = 0 \text{ and } x = 2 \\
 f'(x) &= \text{undefined} \Rightarrow \text{no critical values from this!}
\end{align*}
\]

(b) increasing and decreasing intervals

<table>
<thead>
<tr>
<th>Intervals</th>
<th>([-1,0)]</th>
<th>((0,2))</th>
<th>((2, 3])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Value</td>
<td>(x = \frac{1}{2})</td>
<td>(x = 1)</td>
<td>(x = \frac{5}{2})</td>
</tr>
<tr>
<td>Sign of (f'(x))</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Inc/Dec</td>
<td>INC</td>
<td>DEC</td>
<td>INC</td>
</tr>
</tbody>
</table>

(c) inflection points

\[
\begin{align*}
 f''(x) &= 6x - 6 \\
 f''(x) &= 0 \Rightarrow x = 1 \\
 f''(x) &= \text{undefined} \Rightarrow \text{no inflection points from this!}
\end{align*}
\]

(d) concavity intervals

<table>
<thead>
<tr>
<th>Intervals</th>
<th>([-1,1)]</th>
<th>((1, 3])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Value</td>
<td>(x = 0)</td>
<td>(x = 2)</td>
</tr>
<tr>
<td>Sign of (f''(x))</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Concave up/down</td>
<td>DOWN</td>
<td>UP</td>
</tr>
</tbody>
</table>

(e) all extrema

To find absolute mins/maxs:

\[
\begin{align*}
 f(-1) &= -4 \\
 f(3) &= 0 \\
 f(0) &= 0 \\
 f(2) &= -4
\end{align*}
\]

Relative Max: \(x=0\)
Relative Min: \(x=2\)
Absolute Min: \(x=-1\) and \(x=2\)
Absolute Max: \(x=0\) and \(x=3\)

5. For \( f(x) = x^3 - 3x \),

(a) find the relative extrema using the 1st derivative test.

Critical Values:

\[
\begin{align*}
 f'(x) &= 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1) \\
 f'(x) &= 0 \Rightarrow x = 1 \text{ and } x = -1 \\
 f'(x) &= \text{undefined} \Rightarrow \text{no critical values from this!}
\end{align*}
\]


<table>
<thead>
<tr>
<th>Intervals</th>
<th>$(-\infty, -1)$</th>
<th>$(-1, 1)$</th>
<th>$(1, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Value</td>
<td>$x = -2$</td>
<td>$x = 0$</td>
<td>$x = 1$</td>
</tr>
<tr>
<td>Sign of $f''(x)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>Inc/Dec</td>
<td>INC</td>
<td>DEC</td>
<td>INC</td>
</tr>
</tbody>
</table>

Relative Max: $x = -1$
Relative Min: $x = 1$

(b) verify what you found in part (a) with the 2nd derivative test.
Find the 2nd derivative & test your critical values

$$f''(x) = 6x$$
$$f''(-1) = -6 < 0 \implies x=-1 \text{ is relative MAX}$$
$$f''(1) = 6 > 0 \implies x=1 \text{ is relative MIN}$$

6. Find 2 positive numbers such that the sum of the first and twice the second is 100 and the product is a maximum.

- Equation from the given information: $x + 2y = 100$
- We want to *maximize* their product: $P=xy$
- Getting $x + 2y = 100$ in terms of $x$: $x = 100 - 2y$
- Plugging $x = 100 - 2y$ into $P = xy$: $P = (100 - 2y)y$
- Now maximize $P$ by taking the derivative and setting it equal to zero to find your critical values:

$$P' = 100 - 4y$$
$$P' = 0 \implies y = 25$$

- Check if $y = 25$ is a maximum by using the 1st or 2nd derivative test for relative extrema. I’ll check with the 2nd derivative test:

$$P'' = -4$$
$$P'' < 0 \implies y=25 \text{ is a MAX}$$

- Now solve for the value of $x$: $x = 100 - 2(25) = 50$
7. The combined perimeter of a circle and a square is 16in. Find the dimensions of the circle and square that produce a minimum total area.

- Perimeter of a circle is the circumference: $2\pi r$
- Area of a circle: $\pi r^2$
- Let $x$ be the length of each side of the square.
- Equation from the given information: $16 = 2\pi r + 4x$
- We need to minimize the total area: $A = \pi r^2 + x^2$
- Getting $16 = 2\pi r + 2x$ in terms of $r$: $r = \frac{16 - 4x}{2\pi} = \frac{8 - 2x}{\pi}$
- Plugging $r = \frac{8 - 2x}{\pi}$ into $A = \pi r^2 + x^2$:

$$A = \pi \left(\frac{8 - 2x}{\pi}\right)^2 + x^2$$
$$A = \frac{(8 - 2x)^2}{\pi} + x^2$$

- Now minimize $A$:

$$A' = \frac{2((\pi + 4)x - 16)}{\pi}$$
$$A' = 0 \implies x = \frac{16}{4 + \pi} = 2.24\text{in}$$

- Check if $x = 2.24$ is a minimum:

$$A'' = \frac{2(\pi + 4)}{\pi}$$
$$A'' > 0 \implies x = 2.24 \text{ is a MIN}$$

- Now solve for the value of $r$: $r = \frac{8 - 2 \left(\frac{16}{4 + \pi}\right)}{\pi} = 1.12\text{in}$

8. For $0.3x^2 + 6x + 600$, find the average cost for producing 25 units.

Average Cost: $C(x) = \frac{0.3x^2 + 6x + 600}{x}$
$$C(25) = \frac{0.3(25)^2 + 6(25) + 600}{25} = 37.50$$