Recall that we say that a group $G$ acts on a set $X$ if the following two properties are satisfied.

(i) $e.x = x$ for all $x \in X$.

(ii) $a.(b.x) = (ab).x$ for all $a, b \in G$ and all $x \in X$. (Notice that the multiplication $ab$ is multiplication in $G$, while the “.” multiplication is the group action).

Now fix $x \in X$. We recall that the orbit of $x$, denoted $\text{orb}_G(x)$, is the set $\{y \in X | \exists g \in G \text{ such that } g.x = y\}$. We recall that the stabilizer of $x$, denoted $\text{stab}_G(x)$ is the set $\{g \in G | g.x = x\}$, it is always a subgroup.

We also recall some quick facts about group actions so suppose $G$ acts on $X$.

1. $X$ is the disjoint union of its distinct orbits.

2. $|\text{orb}_G(x)| = [G : \text{stab}_G(x)]$ (the size of an orbit is the same as the number cosets of the stabilizer).

3. If $G$ is finite, then (2) reduces to $|\text{orb}_G(x)| \cdot |\text{stab}_G(x)| = |G|$.

1. Let $G$ be a group and $H$ be a subgroup. Set $X$ to be the set of left cosets of $H$ (notice that $X$ is not necessarily a group because $H$ is not normal). Prove that $G$ acts on $X$ with the following action.

$$g.(aH) = (ga)H.$$ 

**Solution:** First we show that the action is well defined so suppose that $cH = c'H$. Then $g.(cH) = (gc)H = g(cH) = g(c'H) = (gc')H = g.(c'H)$. Now we prove that the two properties of an action, (i) and (ii) above hold. For (i), simply notice that $e.(cH) = (ec)H = cH$. For (ii) observe that $a.(b.(cH)) = a.((bc)H) = (a(bc))H = ((ab)c)H = (ab).(cH)$ as desired.

2. With the notation as in 1., consider $G = S_3$ and $H = \langle (12) \rangle$. Compute the orbits and stabilizers of all the elements of $X$.

**Solution:** First we write down $X$, the left cosets of $H$. Note

$$X = \{\{e, (12)\}, \{(13), (123)\}, \{(23), (132)\}\}.$$ 

We first compute the orbit of $\{e, (12)\}$. Notice that $e.\{e, (12)\} = \{e, (12)\}$, $(13).\{e, (12)\} = \{(13), (123)\}$ and $(23).\{e, (12)\} = \{(23), (132)\}$. Thus $\text{orbs}_3(\{e, (12)\}) = X$. Since the orbits are disjoint, this is the only orbit.

Now we compute the stabilizer of each element.

$$\text{stabs}_3(\{e, (12)\}) = \{e, (12)\}$$

$$\text{stabs}_3(\{e, (13)\}) = \{e, (23)\}$$

$$\text{stabs}_3(\{e, (23)\}) = \{e, (13)\}$$
3. Fix the notation as in 1., and fix \( g \in G \). We define the function \( \tau_g : X \to X \) by the rule.
\[
\tau_g(aH) = (ga)H
\]
Prove that \( \tau_g \) really is a permutation (ie, prove it is bijective).

**Solution:** Consider the function \( \tau_g^{-1} \). I claim that \( \tau_g \circ \tau_g^{-1} = \text{id}_X = \tau_g^{-1} \circ \tau_g \).

First notice that for any \( aH \in X \), \( \tau_g \circ \tau_g^{-1}(aH) = \tau_g((g^{-1}a)H) = (g(g^{-1}a))H = aH \). Likewise \( \tau_g^{-1} \circ \tau_g(aH) = \tau_g^{-1}((ga)H) = (g^{-1}(ga))H = aH \). This proves \( \tau_g \) has an inverse function and thus \( \tau_g \) is a permutation.

4. Fix the notation as in 1.. Recall that \( S_X \) is the set of permutations on \( X \) prove that \( \phi : G \to S_X \) defined by the rule \( \phi(g) = \tau_g \) is group homomorphism. Also prove that the kernel of \( \phi \) is contained within \( H \).

**Solution:** First notice that \( \tau_{gg'}(aH) = ((gg')aH) = (g(g'a))H = \tau_g((g'a)H) = \tau_g \circ \tau_{g'}(aH) \). Since \( aH \) was arbitrary, this proves that \( \tau_{gg'} = \tau_g \circ \tau_{g'} \). Therefore \( \phi(gg') = \tau_{gg'} = \tau_g \circ \tau_{g'} = \phi(g) \circ \phi(g') \) which proves that \( \phi \) is a homomorphism.

Suppose now that \( g \in \ker \phi \). Thus \( \phi(g) = \tau_g \) is the identity. In other words, \( \tau_g(aH) = aH \) for all \( aH \in X \). In particular, \( \tau_g(eH) = gH = H \). Therefore \( g \in H \) and so \( \ker \phi \subseteq H \) as desired.