Definition 0.1. A permutation $\alpha \in S_n$ is called even if it can be written as a product of an even number of transpositions (ie, cycles of the form $(ij)$). A permutation $\alpha \in S_n$ is called odd if it isn’t even.

1. Set $A_n$ to be the set of all even permutations in $S_n$. Prove that $A_n$ is a group with binary operation composition (ie, the induced binary operation from $S_n$).

Solution: First we prove that composition is a binary operation: If $\alpha$ can be written as a product of an even number $n$ of 2-cycles, and $\beta$ can also be written as a product of an even number $m$ of 2-cycles, then $\alpha \beta$ can be written as a product of $n + m$, which is even, 2-cycles. Thus composition is indeed a binary operation.

Now we prove that $A_n$ is indeed a group. Associativity is immediate because function composition is always associative. The identity $e = (12)(12)$ can certainly be written as an even number of two cycles, thus $e \in A_n$. For inverses, suppose that $\alpha = (ab)(cd)\ldots(wx)$ where there are an even number of pairs transpositions. $\alpha^{-1} = (wx)\ldots(ab)$ thus can also be written as an even number of transpositions. Thus $A_n$ is indeed a group.

2. Identify all the elements of $A_2$, $A_3$ and $A_4$. Are any of these groups Abelian?

Solution:

(i) $A_2$. In this case $S_2 = \{e, (12)\}$ and so $A_2 = \{e\}$. This group is certainly Abelian (there is nothing to check).

(ii) $A_3$. Now $S_3 = \{e, (12), (13), (23), (123), (132)\}$. Thus $A_3 = \{e, (123) = (13)(12), (132) = (12)(13)\}$. This group is also Abelian since $(123)(132) = e = (132)(123)$ (note for any $\alpha$, $ae = a = ea$, likewise $\alpha^2 = \alpha\alpha$ – in this last case, the order of $\alpha$ multiplied by itself certainly doesn’t matter).

(iii) $A_4$. I won’t write down $S_4$, but I will note that any $n$-cycle is even if and only if $n - 1$ is even. Note that $(12\ldots n) = (1n)\ldots(12)$ which has $n - 1$ terms in its product. Thus, $A_4 = \{e, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$. This group is not Abelian since $(123)(124) = (13)(24)$ but $(124)(123) = (14)(23)$. 

1
3. Conjecture and prove a formula for the number of elements in $A_n$

*Hint:* Compare the size of $A_2$, $A_3$ and $A_4$ with the size of $S_2$, $S_3$ and $S_4$ respectively. To prove your formula, consider the function from the set of even permutations to the set of odd permutations given by multiplication (on the left) by $(12)$ and show it is bijective.

**Solution:** We first make the assumption that $n \geq 2$, as in the case that $n = 1$, our proposed formula breaks down (in this case $A_n = S_n = \{ e \}$). Our formula is $n!/2$ since $A_2 = 1$ while $S_2 = 2 = 2!$, and $A_3 = 3$ while $S_3 = 6 = 3!$ and $A_4 = 12$ while $S_4 = 24 = 4!$. We now prove that this formula is correct.

Let $B_n = S_n \setminus A_n$. It is sufficient to show that $B_n$ (the set of odd permutations) is the same size as $A_n$ because then the number of elements of $A_n$ is the number of elements of $S_n$ over 2, or $n!/2$.

Consider the function $\phi : A_n \to B_n$ defined by the rule $\phi(\alpha) = (12)\alpha$. We will show that $\phi$ is bijective proving the theorem.

For injectivity, suppose first that $\phi(\alpha) = \phi(\beta)$, thus $(12)\alpha = (12)\beta$ and so $\alpha = (12)(12)\alpha = (12)(12)\beta = \beta$ which proves that $\phi$ is injective.

For surjectivity, choose now $\gamma \in B_n$, $\gamma$ is an odd permutation and so $(12)\gamma$ is even. But now $\phi((12)\gamma) = (12)(12)\gamma = \gamma$ and so $\phi$ is indeed surjective.

Thus $\phi$ is bijective and the proof is completed.

4. Show that a permutation with odd order must always be an even permutation.

**Solution:** Suppose that $\alpha^{2n+1} = e$ for some integer $n$. Writing $\alpha$ as a product of $m$ transpositions, and plugging this into $\alpha^n$, we see that a product of $m(2n+1)$ transpositions is equal to $e$. But in class we showed that $e$ can only be written as a product of an even number of transpositions. Thus $m(2n+1)$ is even and thus $m$ is also even, which proves that $\alpha$ is an even permutation as desired.