In this homework assignment, you will use the following definition of a ring.

**Definition 0.1.** A ring $R$ is a set satisfying the following properties ($a, b, c \in R$ are arbitrary elements).

1. $R$ has two binary operations denoted by $+$ and $\cdot$.
2. $(a + b) + c = a + (b + c)$.
3. There is an element $0 \in R$ such that $0 + a = a = a + 0$.
4. There is an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$.
5. $a(b + c) = ab + ac$.

Often rings are also assumed to satisfy the following (optional) properties.

- $(6^*)$ $a(bc) = (ab)c$. (associativity)
- $(7^*)$ $ab = ba$. (commutativity)
- $(8^*)$ There is an element $1 \in R$ such that $1a = a = a1$. (identity)

All rings below will be assumed to satisfy $(6^*)$, associativity, but not necessarily $(7^*)$ or $(8^*)$, unless specified.

1. Give an example of a finite noncommutative ring. Give an example of an infinite noncommutative ring that does not have a multiplicative identity.
2. Consider the set $\{0, 2, 4\}$ under addition and multiplication modulo 6. Verify that this is a ring with a multiplicative identity.
3. Give an example of a ring $R$ with elements $a, b \in R$ such that the equation $ax = b$ has more than one solution. Give a different example when it has zero solutions.
4. Show that a ring that is a cyclic group under addition is commutative.
5. Suppose that $R$ is a ring and $a, b \in R$. For any integers $m, n \in \mathbb{Z}$, we define $ma$ and $nb$ to be $a$ added to itself $m$-times respectively, $b$ added to itself $n$-times. Find an example of distinct integers $m, n$ and elements of a ring $a, b$ such that $ma = nb$ and $na = nb$ but $a \neq b$. Show this can’t happen if $n$ and $m$ are relatively prime.
6. Let $n > 1$ be an integer. Suppose that $R$ is a ring such that $x^n = x$ for all $x \in R$. Show that $ab = 0$, for any $a, b \in R$ implies then that $ba = 0$. If $n$ is even, prove additionally that $a = -a$ for all $a \in R$.
7. Show that every element of $\mathbb{Z}_{\mod n}$ is either a zero-divisor or a unit.
8. Suppose that $R$ is a finite commutative ring with multiplicative identity, show that every element of $R$ is either a zero-divisor or a unit.
9. Prove that a finite ring $R$ must have positive characteristic.
10. Suppose that $x, y$ belong to a ring of characteristic $p$ where $p$ is prime. Prove that $(x + y)^p = x^p + y^p$. Give an example to show that the equation does not hold if $p = 4$ (and thus is not prime).
11. Find all units, zero-divisors, and nilpotent elements in $\mathbb{Z}_{\mod 3} \oplus \mathbb{Z}_{\mod 6}$ (here the addition and multiplication are both component-wise).
12. Consider the set $\mathbb{Z}_{\mod n}[i] = \{a + bi | a, b \in \mathbb{Z}_{\mod n}\}$ with addition and multiplication defined $\mod n$ and with the relation $i^2 = -1$. Prove that this is not an integral domain if $n = 2, 5$ or 13 but it is an integral domain if $n = 3, 7$ or 11.