For this homework, we assume all rings are commutative, associative and with multiplicative identity. We assume that all homomorphisms send 1 to 1.

1. Show that a ring \( R \) is Noetherian if and only if for every ideal \( I \subseteq R \), there exists elements \( x_1, \ldots, x_n \in I \) such that \( I = (x_1, \ldots, x_n) \). We talked through this in class on Wednesday April 6th, but now write down the details carefully.

2. Suppose that \( R \) is a unique factorization domain. Prove that every irreducible element in \( R \) is prime.

3. Prove or disprove, every subring of a PID is a PID.

4. Same question as #3, but with UFD.

5. Show that \( 1 - i \) is irreducible in \( \mathbb{Z}[i] \).

6. Prove that every non-zero prime ideal in a PID is maximal.

7. Suppose that \( \phi : R \to S \) is a surjective ring homomorphism. Suppose that \( x \in R \) is an irreducible element. Is it true that \( \phi(x) \) is also irreducible? Prove it or give a counter-example.

8. Show that a non-constant polynomial from \( \mathbb{Z}[x] \) that is irreducible (as an element of \( \mathbb{Z}[x] \)) is primitive.

9. Show that \( x^4 + 1 \in \mathbb{Q}[x] \) is irreducible. But \( x^4 + 1 \in \mathbb{R}[x] \) is reducible (not irreducible).

10. Explicitly construct a field with 49 elements.

11. Determine which of the following polynomials below are irreducible over \( \mathbb{Q} \) (ie, irreducible elements of \( \mathbb{Q}[x] \)).

(a) \( x^5 + 9x^4 + 12x^2 + 6 \)
(b) \( x^4 + x + 1 \)
(c) \( x^4 + 3x^2 + 3 \)
(d) \( x^5 + 5x^2 + 1 \)
(e) \( \frac{7}{2}x^5 + \frac{9}{2}x^4 + 15x^3 + \frac{3}{2}x^2 + 6x + \frac{3}{14} \)