1. Periodic solutions of Liénard type equations

This note is devoted to alternate (but very similar) proof of Theorem 9 in Chapter VII of the lecture notes. In particular we shall employ the note on Schaefer’s theorem given among the notes. This note is to be read in the context of Chapter VII and the assumptions made are those made there and the numbering is that used there. Recall that if the linear equation
\[ u' = A(t)u \]
is nonresonant, then \( u \) is a \( T \)-periodic solution of
\[ u' = A(t)u + f(t, u), \]
if and only if \( u \in E := \{ v \in C([0, T], \mathbb{R}^N) : u(0) = u(T) \} \) is a fixed point of the operator \( S \circ f \) defined by formula (7) (replace \( S \) by \( S \circ f \)). We rewrite this as
\[ (S \circ f)(u)(t) = \int_0^T G(t, s)f(s, u(s))ds, \]
as was done during the lectures. There we showed also that \( S \circ f \) is a completely continuous mapping of \( E \) to itself. We also note that
\[ S \circ (\epsilon f) = \epsilon S \circ f, \]
for any real number \( \epsilon \). Hence we may apply Schaefer’s theorem to the fixed point problem
\[ u = \epsilon S \circ f(u). \]

We shall apply these ideas to prove the existence of periodic solutions of Liénard type oscillators of the form
\[ x'' + h(x)x' + x = e(t), \tag{1.1} \]
where
\[ e : \mathbb{R} \rightarrow \mathbb{R} \]
is a continuous \( T \)-periodic forcing term and
\[ h : \mathbb{R} \rightarrow \mathbb{R} \]
is a continuous mapping. We shall prove the following result.

**Theorem 1.1.** Assume that \( T < 2\pi \). Then for every continuous \( T \)-periodic forcing term \( e \), equation (1.1) has a \( T \)-periodic response \( x \).

We note that, since, aside from the continuity assumption, nothing else is assumed about \( h \), we may, without loss in generality, assume that \( \int_0^T e(s)ds = 0 \), as follows from the substitution
\[ y = x - \int_0^T e(s)ds. \]

We hence shall make that assumption. In order to apply our earlier results, we convert (1.1) into a system as follows:
\[ \begin{align*}
  x' &= y \\
  y' &= -h(x)y - x - e(t),
\end{align*} \tag{1.2} \]
and put

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(t, u) = \begin{pmatrix} y \\ -h(x)y - e(t) \end{pmatrix}.$$  

We hence may rewrite the second order equation as

$$u' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u + f(t, u).$$

We note that the linear equation

$$u' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u$$

is nonresonant, since all of its solutions are given by

$$u(t) = \begin{pmatrix} a \sin t + b \cos t \\ a \cos t - b \sin t \end{pmatrix}$$

which are periodic of period $2\pi$ and $T < 2\pi$. We next shall show that Schaefer’s theorem (more precisely the homotopy invariance property of Leray-Schauder degree) may be applied by providing a priori bounds for solutions of the fixed point equation

$$(1.4) \quad u = \epsilon S \circ f(u),$$

for $0 \leq \epsilon \leq 1$ for $f$ given as above, then the result will follow. Now

$$u = \begin{pmatrix} x \\ y \end{pmatrix},$$

is a solution of (1.4) whenever $x$ satisfies

$$(1.5) \quad x'' + x + eh(x)x' = \epsilon e(t).$$

Integrating (1.5) from 0 to $T$, we find that

$$\int_0^T x(s)ds = 0.$$ 

Multiplying (1.5) by $x$ and integrating we obtain

$$(1.6) \quad -\|x'\|_{L^2}^2 + \|x\|_{L^2}^2 = \epsilon \langle x, e \rangle_{L^2},$$

where $\langle x, e \rangle_{L^2} = \int_0^T x(s)e(s)ds$. Now, since (here we use properties of Fourier series)

$$(1.7) \quad \|x\|_{L^2}^2 \leq \frac{T^2}{4\pi^2} \|x'\|_{L^2}^2,$$

we obtain from (1.6)

$$(1.8) \quad \left(1 - \frac{T^2}{4\pi^2}\right) \|x'\|_{L^2}^2 \leq -\epsilon \langle x, e \rangle_{L^2},$$
from which follows that
\[ \|x'\|_{L^2} \leq \left( \frac{2\pi T}{4\pi^2 - T^2} \right) \|e\|_{L^2}, \]
from which, in turn, we obtain
\[ \|x\|_{\infty} \leq \sqrt{T} \left( \frac{2\pi T}{4\pi^2 - T^2} \right) \|e\|_{L^2}, \]
providing an a priori bound on \( \|x\|_{\infty} \). We let
\[ \sqrt{T} \left( \frac{2\pi T}{4\pi^2 - T^2} \right) \|e\|_{L^2} = M, \]
\[ q = \max_{|x| \leq M} |h(x)|, \quad p = \|e\|_{\infty}. \]
Then
\[ \|x''\|_{\infty} \leq q \|x'\|_{\infty} + M + p. \]
Hence, by Landau's inequality (Exercise 6, below), we obtain
\[ \|x'\|^2_{\infty} \leq 4M(q\|x'\|_{\infty} + M + p), \]
from which follows a bound on \( \|x'\|_{\infty} \) which is independent of \( \epsilon \), for \( 0 \leq \epsilon \leq 1 \). These considerations complete the proof Theorem 1.1.