MATH 3150 - EXAM II

SOLUTIONS:

1) (5 pt) Find a harmonic function \( u(x, y) \) on the square \([0, 1] \times [0, 1]\) subject to the following boundary conditions: The function \( u \) is linear along all four edges of the rectangle, and \( u(0, 0) = 3, u(1, 0) = 5 \) \( u(0, 1) = 7 \) and \( u(1, 1) = 11 \). Recall that a function \( u \) is harmonic if

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]

Hint: look for a function of the type \( u(x, y) = axy + bx + cy + d \).

Solution: \( 2xy + 2x + 4y + 3 \).
2) (6 pt) Consider the equation of an oscillating string of length $L = 1$
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial y^2}.
\]
This equation has a unique solution such that the initial velocity is 0 and the initial position is $u(x, 0) = f(x)$ where
\[
f(x) = \begin{cases} 
6x & \text{if } 0 \leq x \leq 1/3 \\
3 - 3x & \text{if } 1/3 \leq x \leq 1.
\end{cases}
\]
Use the D’Alembert method to graph $u(x, 1/2)$.

Solution:
\[
u(x, 1/2) = \begin{cases} 
-3x & \text{if } 0 \leq x \leq 1/6 \\
3x/2 - 3/4 & \text{if } 1/6 \leq x \leq 5/6 \\
3 - 3x & \text{if } 5/6 \leq x \leq 1.
\end{cases}
\]
3) (6 pt) Compute the first two non-zero terms of the solution of the heat equation with $L = \pi$, the boundary conditions $u(0, t) = u(\pi, t) = 0$ and initial condition $u(x, 0) = f(x)$ where

$$f(x) = \begin{cases} 
x & \text{if } 0 \leq x \leq \pi/2 \\
x - \pi & \text{if } \pi/2 \leq x \leq \pi.
\end{cases}$$

Solution. The general form of the solution is $u(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-tn^2}$ where

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) \, dx.$$ 

Since $f(x)$ is odd with respect to $x = \pi/2$, $b_n = 0$ for $n$ odd and

$$b_n = \frac{4}{\pi} \int_{0}^{\pi/2} x \sin(nx) \, dx$$

for $n$ even. Integrating by parts,

$$\int_{0}^{\pi/2} x \sin(nx) \, dx = \left(-x \frac{\cos(nx)}{n}\right)_{0}^{\pi/2} + \int_{0}^{\pi/2} \frac{\cos(nx)}{n} \, dx = \frac{\pi \cos(n\pi/2)}{2n}.$$ 

(The second integral is 0, using $\sin(n\pi/2) = 0$ for $n$ even!) Substituting $n = 2$ and $n = 4$ gives the first two terms:

$$u(x, t) = \sin(2x)e^{-4t} - \frac{1}{2} \sin(4x)e^{-16t^2} + \ldots$$

NB: graph $\sin(2x) - \frac{1}{2} \sin(4x)$ to see how it neatly approximates the function $f(x)$. 

4) (5 pt) Find the indicial roots $r_1 \geq r_2$ of the regular singularities differential equation

$$2xy'' + (1 + x)y' + y = 0$$

and then (3 pt) find the coefficient $a_1$ of the unique solution in a power series form starting with $x^{r_1}$:

$$y = x^{r_1} + a_1 x^{r_1+1} + \ldots$$

Solution: The power series of 4 summands on the left hand side of the differential equation are

$$2xy'' = 2r(r-1)x^{r-1} + 2a_1(r+1)x^r +$$
$$y' = rx^{r-1} + a_1(r+1)x^r +$$
$$xy' = 0x^{r-1} + rx^r +$$
$$y = 0x^{r-1} + x^r +$$

The coefficients in front of $x^{r-1}$ have to add up to 0. This gives $0 = 2r(r-1) + r = r(2r-1)$, thus indicial roots are $r_2 = 0$ and $r_1 = 1/2$. Next, the coefficients in front of $x^r$ have to add up to 0. This gives

$$0 = 2a_1 r(r+1) + a_1(r+1) + r + 1$$

which, after substituting $r = 1/2$, implies $a_1 = -1/2$. 