3. Metric Spaces

**Definition.** A function $\rho : X \times X \to \mathbb{R}_+$ is a **metric** if and only if it satisfies these conditions:

a) $\rho(x, y) = 0$ if and only if $x = y$ (otherwise $\rho(x, y) > 0$),

b) $\rho(x, y) = \rho(y, x)$,

c) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z$ in $X$.

**Remark.** For two sets $X, Y$, the product set $X \times Y$ is the set of all pairs $\{(x, y); x \in X, y \in y\}$. Suppose $I$ is a set, and to each $i \in I$ we have associated a set $X_i$. Then the product $\times_{i \in I} X_i$ is the set of all functions $\phi : I \to \bigcup_{i \in I} X_i$ such that $\phi(i) \in X_i$ for all $i$.

**Examples 3A, 3B, 3C.** Generalize 3C to $\mathbb{R}^n$. This is the **Euclidean metric**.

**Problem:** **13.** Look at example 3.1. Let $L^\infty$ be the set of all bounded sequences of real numbers. For $s$ in $L^\infty$, let $|s|_\infty$ be the least upper bound of the set $|s_n|$. Define $\rho_\infty(s, s') = |s - s'|_\infty$. Show that $\rho$ is a metric.

Look at 3.2. Generalize this to the set $L^1$ of absolutely convergent series.

**Definition.** Let $(X, \rho)$ be a metric space. The sets

$$D_r(a) = \{x \in X; \rho(a, x) < r\}$$

$$D_r[a] = \{x \in X; \rho(a, x) \leq r\}$$

$$S_r(a) = \{x \in X; \rho(a, x) = r\}$$

are the **open ball**, **closed ball**, **sphere** with center $a$ and radius $r$.

**Problem 14.** For $\mathbb{R}^2$, describe these sets for each of the metrics 3C,3.1,3.2.

**Proposition.** Let $(X, \rho)$ be a metric space. The sets $D_r(a)$ form a base for a topology on $X$, called the **metric topology**.

**Problem **15.** Prove the above proposition. Show that the sets $D_r[a]$, $S_r(a)$ are closed in this topology.

**Definition.** Let $A$ be a subset of a metric space $(X, \rho)$. Then $(A, \rho|_A)$ is also a metric space. We say that $A$ is a **subspace** of $X$. 
**Problem 16.**

a). Let \( A = [0, 1] \cup [2, 3] = \{ x \in \mathbb{R}; 0 \leq x \leq 1 \text{ or } 2 \leq x \leq 3 \} \) as a subspace of \( \mathbb{R} \). Show that the subset \([0,1]\) is open in \( A \).

b). Let \( A = (0, 1) \cup (1, 2) = \{ x \in \mathbb{R}; 0 < x < 1 \text{ or } 1 < x < 2 \} \) as a subspace of \( \mathbb{R} \). Show that the subset \((0,1)\) is closed in \( A \).

**Problem 17.** Let \( C \) be the Cantor set, considered as a subspace of \( \mathbb{R} \). Let \( a \) be in \( C \), and \( \epsilon > 0 \). Show that there is an \( r < \epsilon \) such that \( D_r(a) = D_r[a] \); that is, the ball \( D_r(a) \) is both open and closed.

**Definition.** A subset \( A \) of a metric space \( (X, \rho) \) is **bounded** if there is a number \( M \) such that \( \rho(x, y) \leq M \) for all \( x, y \) in \( A \). The least upper bound of numbers \( M \) satisfying this condition is the **diameter** of \( A \).

**Problem 18.** Show that a set \( U \) in a metric space \( (X, \rho) \) is open if and only if it satisfies the condition: for every \( a \in U \), there is an \( r > 0 \) such that \( D_r(a) \subseteq U \).

**Definition.** Let \( X \) be a set, and \( \rho_1 \) and \( \rho_2 \) two metrics on \( X \). The metrics are said to be **equivalent** if they determine the same topology on \( X \).

**Problem 19.**

a) Show that two metrics on a set \( X \) are equivalent if and only if for every \( a \in X \), every ball centered at \( a \) in one metric contains a ball centered at \( a \) in the other metric.

b). Can you prove (or disprove) this: Two metrics on a set \( X \) are equivalent if and only if every ball in one metric contains a ball in the other metric.

**Problem 20.** 3.29 and/or:

\( L_1 \) is a subset of \( L_\infty \) since every absolutely convergent sequence is bounded. So \( L_1 \) has its natural topology, as defined above, and also that as a subspace of \( L_\infty \). These topologies are not equivalent. In both cases, one topology is finer than the other; which is it?

4. **Subspaces**

**Definition.** Let \( X \) be a topological space, and \( A \) a subset of \( X \). The **relative topology** on \( A \) is given by \( \{ A \cap U; U \text{ open in } X \} \) as the set of open sets.

**Problem 21.** Show that if \( (X, \rho) \) is a metric space and \( A \) a subset of \( X \), then the topology induced by the metric \( \rho|_A \) is the relative topology.

**Problem 22.** 4E and 4F.

5. **Position of a Point in a Set**

**Definition.** Let \( x \) be a point in a subset \( A \) of a topological space \( X \).
$x$ is an **interior point** of $A$ if there is an open set $U$ with $x \in U \subset A$. The set of interior points of $A$ is denoted int$A$.

$x$ is an **exterior point** of $A$ if there is an open set $U$ with $x \in U \subset X - A$. The set of exterior points of $A$ is denoted ext$A$.

$x$ is a **boundary point** of $A$ if every open set $U$ with $x \in U$ intersects both $X$ and $X - A$. The set of interior points of $A$ is denoted $\partial A$.

**Definition** $x$ is a **limit point** of a set $A$ if every open set containing $x$ intersects $A - \{x\}$.

$x$ is a **isolated point** of $A$ if there is an open set $U$ such that $U \cup A = \{x\}$.

Show that a set is closed if and only if it contains all its limit points. This will help in some of the following problems.

**Problem** **23.** Show that, for any set $A$, $X$ is the disjoint union of int$A$, ext$A$, $\partial A$. Show that int$A$ and ext$A$ are open, and $\partial A$ is closed.

**Problem 24.** a) Show that int$A$ is the maximal open set contained in $A$, and is the union of all open sets contained in $A$. b) Show that int$(X - A) = $ ext$A$.

**Definition.** Given a set $A$ in a topological space $X$, the **closure** of $A$, denoted $\overline{A}$ is the intersection of all closed sets containing $A$.

Show that, for $A$ a subset of a topological space $\overline{A} = \text{int}A \cup \partial A$.

**Definition.** Let $A$ be a subset of a topological space $X$. $A$ is **everywhere dense** if its closure is $X$. A set is **nowhere dense** if its closure has empty interior.

Note that this is different from the definition in the text, and conforms to common usage.

**Problem** **25.** $A$ is everywhere dense in $X$ if and only if $A$ intersects every open set. A closed set is nowhere dense if and only if it is equal to its boundary.

**Definition.** A topological space is said to be **Hausdorff** if this condition is satisfied; for $x \neq y$ there are disjoint open sets $U$ and $V$ with $x \in U$ and $y \in V$.

**Problem 26.** Of the examples considered so far, which are not Hausdorff?

**Remark** Two things to note; the definition of ”nowhere dense” in the text is not the common definition, so stay with the one in the supplementary notes. At the end of those notes, I introduced the concept of ”Hausdorff”. This is a regularity assumption which I shall assume from now on; so if I say ”topological” space, I mean a Hausdorff space. All metric spaces are Hausdorff.

6. Maps
This section contains some concepts and notation with which it is necessary to become familiar.

**Definition** A mapping $f$ of a set $X$ to a set $Y$ (written $f : X \to Y$) is a rule which assigns to each element of $X$ exactly one element of $Y$. Other words: map, function.

Now, the word “rule” is an intuitive notion which can be eliminated by the following completely set-theoretic definition. Essentially, we identify the mapping $f$ with its graph. To make this clear, let us call, for any $x \in X$, the subset $\{x\} \times Y$ of $X \times Y$ the section $\sigma_x$, and for any $y \in Y$, the subset $X \times \{y\}$ of $X \times Y$ the section $\sigma_y$.

**Definition** A mapping $f$ of a set $X$ to a set $Y$ is a subset $G_f$ of $X \times Y$ with this property: for every $x \in X$, $G_f \cap \sigma_x$ consists of just one point. We refer to that point as $(x, f(x))$.

The identity $I : X \to X$ is defined by $G_I = \{(x, x) \in X \times X; x \in X\}$.

Now, we elaborate:

**Definition** Given a mapping $f$ of a set $X$ to a set $Y$, the range of $f$ is the set of $y \in Y$ such that $G_f \cap \sigma_y$ is nonempty. $f$ is surjective (or onto) if its range is $Y$. $f$ is injective if, for all $y$ in the range of $f$, $G_f \cap \sigma_y$ consists of just one point.

**Definition.** Given $f : X \to Y$ and $g : Y \to Z$, we define the composition $h = g \circ f$ by the subset $G_h = \{(x, g(f(x)); x \in X\}$ of $X \times Z$.

**Definition.** If $f : X \to Y$ is both injective and surjective, then $G_f$, considered as a subset of $Y \times X$, defines a mapping from $Y$ to $X$ called the inverse of $f$, and denoted $f^{-1}$. We say that $f$ (and of course, also $f^{-1}$) is an invertible mapping.

Verify that $f \circ f^{-1} = I$. $f^{-1} \circ f = I$.

Now that we are done with those formalities, we’ll use the “rule” conception of a function.

**Definition.** Let $f : X \to Y$. For $A \subset X$, $f(A) = \{f(x); x \in A\}$. For $B \subset Y$, $f^{-1}(B) = \{x \in X; f(x) \in B\}$.

7. Continuous Maps

Of course, the whole point of topology is to be able to discuss, with good foundation, the concepts of continuity and convergence. We are finally getting there.

**Definition.** Let $f : X \to Y$, where $X$ and $Y$ are topological spaces. We say that $f$ is continuous if, for every open set $U$ in $Y$, $f^{-1}(U)$ is open in $X$.

**Problem**: 27. (7A). $f : X \to Y$ is continuous if and only if for every closed set $C$ in $Y$, $f^{-1}(C)$ is closed in $X$. 
Problem: **28. Do 7.2

**29. Let \( \mathcal{F} \) be a collection of maps \( f : X \to Y \) be any map, with \( Y \) a topological space. Describe the coarsest topology on \( X \) for which the collection \( \mathcal{F} \) consists of continuous maps.

Remark. If \( X \) is a topological space and we take \( \mathcal{F} \) as the collection of real-valued functions, then the topology defined by Problem 28 is coarser than the given topology. If the space is Hausdorff and the topologies coincide, then \( X \) is said to be a **regular** topological space. We’ll return to this notion.

Problem: 30. (7E). The composition of continuous maps is continuous.

Problem: 31. Do 7.6

Problem: 32. Do 7.10 and 7.13

Problem: 32. Let \( f : X \to Y \) be a mapping, with \( X \) a topological space and \( Y \) a metric space with metric \( \rho \). Show that \( f \) is continuous if and only if, for every \( a \in Y \) and \( r > 0 \), \( f^{-1}(D_r(a)) \) is open in \( X \).

**Local Continuity**

**Definition.** Let \( f : X \to Y \), where \( X \) and \( Y \) are topological spaces. For \( a \in X \), we say that \( f \) is **continuous at** \( a \) if, for every open set \( U \) containing \( f(a) \), \( f^{-1}(U) \) contains an open set containing \( a \). (Verify that this is the same as the definition in the text).

Problem: 33. (7I) \( f \) is continuous if and only if it is continuous at every point of \( X \).

Problem: 34. The above definition is equivalent to the \( \epsilon - \delta \) definition. Precisely, let \( f : X \to Y \), where \( X \) and \( Y \) are metric spaces. Then, for \( a \in X \), \( f \) is continuous at \( a \) if and only if, for every \( \epsilon > 0 \),
\[
\rho_X(x, a) < \delta \quad \text{implies} \quad \rho_Y(f(x), f(a)) < \epsilon .
\]

Problem: 35. For \( X \) a topological space, let \( C(X) \) be the set of real-valued functions on \( X \). Show that for \( f, g \in C(X) \), \( f + g, fg, \min(f, g), \max(f, g) \) are all in \( C(X) \). Also, if \( g(a) \neq 0 \), \( f/g \) is continuous at \( a \).

**Definition.** Let \( X \) and \( Y \) be topological spaces, and \( f : X \to Y \) am invertible mapping. \( f \) is a **homeomorphism** if both \( f \) and \( f^{-1} \) are continuous.

Equivalently, an invertible map \( f : X \to Y \) is a homeomorphism if and only if \( f \) is continuous and, for every open set \( U \subset C \), \( f(U) \) is open in \( Y \).

Problem: **36. Show that, for \( S^2 = S_1((0,0,0) \) in \( \mathbb{R}^3 \), and \( P \) any point in \( S^2 \), that \( S^2 - \{ P \} \) is homeomorphic to \( \mathbb{R}^2 \).**