SLCSE Math 1050, Spring, 2013
Lesson 2, Monday, January 14, 2013: Sequences

Note: The "activities" are to be done and discussed in class. Homework, due at 4 pm Monday, Jan 28, 2013 (Monday, Jan 21, 2013 is a holiday) consists of all problems (designated by the word **Problem**),unless otherwise stated.

Update on Homework

Due today, Jan 14: Problems 1-7 of Lesson 1;

No Class Monday, January 14, Martin Luther King Day;

Due in two weeks, Jan 28: Problems 8-13 of Lesson 1 and Problems 1-9 of Lesson 2.

Definitions and Examples

A *sequence* is a list of numbers. That is, there is a first number, a second number, and for every natural number n an nth number. Sequences are written in the form

$$a_1, a_2, a_3, a_4, \dots\ ,$$

where the ... means that the pattern goes on forever. Sometimes the pattern is clear:

$$1, 2, 3, 4, \dots$$

is telling us that, for any n, the nth term is n. Sometimes the pattern is not so clear:

$$4, 14, 42, 59, 86, 125, \dots$$

These are the numbered stops of the Lexington Avenue express in New York. In fact, this is not a sequence: there is no nth stop for any n. So, besides being undecipherable, this was a trick. We'll refrain from tricks, and make the sense of the sequence clear. For example,

$$1, 3, 6, 10, 15, \dots$$

may not be so clear, so instead we'll write

$$1\ ,\ 1+2\ ,\ 1+2+3\ ,\ 1+2+3+4\ ,\ 1+2+3+4+5\ ,\ \dots\ .$$

Of course, the most clear way of identifying a sequence is by an explicit formula for the nth term. Example:

$$6, 12, 20, 30, 42, \dots$$

is not so terribly clear, but this is:

$$6, 12, 20, 30, 42, \dots, n^2 + 3n + 2, \dots\ . \tag{1}$$

Another way of describing a sequence is to explain how, once we have reached a number in a sequence, to calculate the next one. This is called the *recursive* definition of the sequence (and a powerful tool in the use of computers in mathematics). So, sequence (1) could have been described this way:

$a_1 = 6$. Once we have a_n, we define $a_{n+1} = a_n + 2(n+2)$.

So, here $a_2 = a_1 + 2(1+2) = 6 + 2(3) = 12$, and $a_3 = a_2 + 2(2+2) = 12 + 8 = 20$, and so forth.

$n!$ (called n *factorial*) is defined in two ways:
a) For any n, $n!$ is the product of the first n natural numbers.
b) For any n, $n! = n(n-1)!$.

Basic Classes of Sequences

An *arithmetic sequence* is a sequence $a_1, a_2, a_3, \ldots$ where there is some number d such that $a_{n+1} = a_n + d$ for every natural number n. In other words, a sequence $\{a_n\}$ is arithmetic, if the differences $a_n - a_{n-1}$ of successive terms are all the same.

Examples

a) $\{-7, -1, 5, 11, 17, \ldots\}$ is (the beginning of) an arithmetic sequence with $d = 6$.
b) $\{-0, 1, 3, 6, 10, \ldots\}$ is not an arithmetic sequence because the difference between successive terms changes.

A **geometric sequence** s a sequence $a_1, a_2, a_3, \ldots$ where there is some number r such that $a_{n+1} = ra_n$ for every natural number n.

c) $\{5, 10, 20, 40, 80, \ldots\}$ is a geometric sequence with $r = 2$.
d) $\{1, -1, 1, -1, 1, \ldots\}$ is a geometric sequence with $r = -1$.
e) $\{100, 50, 25, 12.5, 6.25, \ldots\}$ is a geometric sequence with $r = 1/2$.
f) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ is a geometric sequence with $r = 1/2$.

Note that the definition of arithmetic and geometric sequences is recursive in the sense that the rule tells how to get from one term to its successor. So, the initial term always has to be made explicit, as examples e) and f) show: they have the same ratio, but start in different places. Sometimes we have to specify several initial terms to define the sequence unambiguously.

The *Fibonacci* sequence: $a_k = a_{k-1} + a_{k-2}$. To get started, we have to specify the first two terms:

$$1, 1, 2, 3, 5, 8, 13, \ldots \qquad 7, -1, 6, 5, 11, 16, \ldots \ .$$

These are both Fibonacci sequences, (the first is the one studied by Fibonacci); and most generally:

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, \ldots \ .$$

Formula for the general term of a sequence

When a sequence is given recursively, in principle, we cannot find the 100th term until we know the 99th, and so forth. However, it may be possible to find a formula that will tell us that; indeed, what the nth term is for any n. Sometimes this is easy, but often it is quite hard.

Problems 1-8

1. What is the formula for the nth term of the sequence $1, 2, 3, 4, 5, \ldots$?

2. What is the formula for the nth term of the sequence $1, 4, 9, 16, 25, \ldots$?

3. What is the formula for the nth term of the arithmetic sequence $2, 6, 10, 14, 18, \ldots$?

4. What is the 232nd term of the sequence in problem 3?

5. What is the formula for the nth term of the arithmetic sequence $a, a + d, a + 2d, a + 3d, \ldots$?

6. What is the formula for the nth term of the sequence $1, -1, 1, -1, 1, -1, \ldots$?

7. What is the formula for the nth term of the sequence $3, 6, 12, 24, 48, \ldots$?

8. What is the formula for the nth term of the sequence $a, ar, ar^2, ar^3, , \ldots$?

Formula for sums of arithmetic and geometric sequences

Given a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ we can create a new sequence by summing the terms of the sequence:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots \ .$$

This sequence is sometimes called the *series* whose general term is a_n, meaning precisely this: the n term is the sum of the first n terms of the a_n sequence. Letting S_n represent the nth term of the series, we can write

$$S_n = a_1 + a_2 + \cdots + a_n = \Sigma_{i=1}^{n} a_i \ ,$$

or, recursively: $S_1 = a_1$, $S_n = S_{n-1} + a_n$. Can we find a formula for the nth term of the series? In general, the answer is usually no, but for arithmetic and geometric series, it can be done, by special tricks more than 2000 years old.

Sums of Arithmetic Series

Consider the sequence of natural integers: $1, 2, 3, 4, 5, \ldots$. The series S_n formed of this sequence has as its nth term the sum of the first n integers. These are called triangular numbers; why?

Can you find a formula for S_n? Here is a trick that works: first, consider an $(n + 1) \times (n + 1)$ square: meaning a square made up of $n+1$ boxes on the horizontal and $n+1$ boxes on the vertical. There are $(n+1)^2$ boxes. The diagonal has $n+1$ boxes, and when the diagonal is taken out, there are two arrays of S_n boxes each. Thus:

$$(n + 1)^2 - (n + 1) = 2S_n \ .$$

Problem 9. Solve for S_n.