1. The height of a right circular cylinder is increasing at a rate of 2 cm/second and the radius in decreasing at a rate of 3 cm/sec. At what rate is the volume changing if the radius is 10 cm and the height is 15 cm? (The formula for the volume of a right circular cylinder is \( V = \pi r^2 h \).)

Solution: Letting \( V(t) \) denote the volume at time \( t \), we use the chain rule to express \( \frac{dV}{dt} \) in terms of \( \frac{dr}{dt} \) and \( \frac{dh}{dt} \) together with the partial derivatives of \( V \). We have

\[
\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt}.
\]

Using the formula \( V = \pi r^2 h \), this becomes

\[
\frac{dV}{dt} = (2\pi rh) \frac{dr}{dt} + (\pi r^2) \frac{dh}{dt}.
\]

At the given time we have \( r = 10, h = 15, \frac{dr}{dt} = -3 \) and \( \frac{dh}{dt} = 2 \), so

\[
\frac{dV}{dt} = (2\pi(10)(15))(-3) + (\pi 10^2)(2) = -900\pi + 200\pi = -700\pi.
\]

Thus the volume is decreasing at the rate of \( 700\pi \) cubic centimeters per second.

2. Suppose that \( f(x, y, z) \) is a function whose gradient is \( 3\mathbf{i} - \mathbf{j} + 5\mathbf{k} \) at the point \((0, 0, 1)\). Find the derivative of the composite function \( f(\sin t, -3t, e^t) \) at \( t = 0 \). (Notice that when \( t = 0 \), \((\sin t, -3t, e^t) = (0, 0, 1)\).)

Solution: By the chain rule, the derivative of the composite function is

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.
\]

The partial derivatives are the components of the gradient at the point, which are 3, -1, and 5. The derivatives of \( x = \sin t, y = -3t, \) and \( z = e^t \) are \( \cos t, -3, \) and \( e^t \) respectively. At \( t = 0 \) these are 1, -3, and 1. Thus

\[
\frac{df}{dt} = (3)(1) + (-1)(-3) + (5)(1) = 3 + 3 + 5 = 11.
\]
3. Let \( f(x, y) = xy^2 - 6x^2 - 3y^2 \). This function has three critical points. Find their coordinates.

**Solution:** Set the partial derivatives with respect to \( x \) and \( y \) equal to zero:

\[
\frac{\partial f}{\partial x} = y^2 - 12x = 0 \\
\frac{\partial f}{\partial y} = 2xy - 6y = 0.
\]

Solving the second equation, we get that \( y = 0 \) or \( x = 3 \). If \( y = 0 \), then the first equation gives that \( 12x = 0 \), so \( x = 0 \). This produces one critical point, \((0, 0)\). If \( x = 3 \), then the first equation is \( y^2 - 36 = 0 \), or \( y^2 = 36 \), so \( y = \pm 6 \). This gives the other two critical points, \((3, 6)\) and \((3, -6)\).

4. Find if each of the critical points in problem 3 is a local minimum, a local maximum, or a saddle point.

**Solution:** Compute the second partial derivatives:

\[
\frac{\partial^2 f}{\partial x^2} = -12, \quad \frac{\partial^2 f}{\partial x \partial y} = 2y, \quad \frac{\partial^2 f}{\partial y^2} = 2x - 6.
\]

This gives that the discriminant is

\[
D = (-12)(2x - 6) - (2y)^2 = -24x + 72 - 4y^2.
\]

At \((0, 0)\), \( D = 72 > 0 \), and since \( \frac{\partial^2 f}{\partial x^2} = -12 \) is negative, this is a local maximum.

At \((3, 6)\) or \((3, -6)\), \( D = (-24)(3) + 72 - 4(36) = -72 + 72 - 144 = -144 < 0 \) so it is a saddle point.

5. Use Lagrange’s method to find the maximum and minimum values of \( 2x - y \) on the circle \( x^2 + y^2 = 1 \).

**Solution:** Use the formula \( \nabla f = \lambda \nabla g \), where \( f(x, y) = 2x - y \) and \( g(x, y) = x^2 + y^2 - 1 \) to get

\[
2i - j = \lambda(2xi + 2yj).
\]

This gives the equations

\[
2x\lambda = 2 \quad \text{and} \quad 2y\lambda = -1.
\]

Solve the first for \( \lambda \) to get \( \lambda = 1/x \), then substitute in the second to get \( 2y/x = -1 \), or \( x = -2y \).

Then substitute this expression into the equation \( x^2 + y^2 = 1 \) to get \( 4y^2 + y^2 = 1 \), which has solutions \( y = 1/\sqrt{5} \) and \( y = -1/\sqrt{5} \). The corresponding solutions for \( x \) are \( x = -2/\sqrt{5} \) and \( x = 2/\sqrt{5} \), so the two points are \((2/\sqrt{5}, -1/\sqrt{5})\) and \((-2/\sqrt{5}, 1/\sqrt{5})\).
Computing $2x - y$ at these points gives $5/\sqrt{5}$ and $-5/\sqrt{5}$, so the value at $(2/\sqrt{5}, -1/\sqrt{5})$ is a maximum and that at $(-2/\sqrt{5}, 1/\sqrt{5})$ is a minimum.

6. Compute the following iterated integrals:

   a. $\int_1^2 \int_0^4 (x+y)^{-2} \, dx \, dy$

   b. $\int_0^2 \int_{y^2}^{y^3} xy \, dx \, dy$

**Solution: a.**

\[
\int_1^2 \int_0^4 (x+y)^{-2} \, dx \, dy = \int_1^2 \left( -(x+y)^{-1} \right)^4_0 \, dy = \int_1^2 \left( -\frac{1}{y+4} + \frac{1}{y} \right) \, dy
\]

\[= (-\ln(y+4) + \ln(y)) \bigg|_1^2 = -\ln 6 + \ln 2 + \ln 5 - \ln 1 = \ln(5/3).\]

b. $\int_0^2 \int_{y^2}^{y^3} xy \, dx \, dy = \int_0^2 \left( \frac{x^2 y^3}{2} \right)_{y^2}^{y^3} \, dy$

\[\frac{1}{2} \int_0^2 \left( y^3 + 6y^2 + 9y - y^5 \right) \, dy = \frac{1}{2} \left( \frac{y^4}{4} + 2y^3 + 9y^2/2 - y^6/6 \right)_{0}^{2}
\]

\[= \frac{1}{2} (4 + 16 + 18 - 32/3) = 82/6 = 41/3.\]

7. Sketch the region over which the integral is taken in problem 6.b.

**Solution:** The picture should show the region surrounded by the $x$-axis from $(0, 0)$ to $(3, 0)$, the line $x = y + 3$ from $(3, 0)$ to $(5, 2)$, a horizontal line from $(5, 2)$ to $(4, 2)$, and finally the parabola $x = y^2$ from $(4, 2)$ back to $(0, 0)$.

8. Find the integral $\iiint_R (4x^2y) \, dA$, where $R$ is the region inside the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$.

**Solution:**

The integral can be set up on either order and will be either

\[\int_0^1 \int_0^y 4x^2 y \, dx \, dy \quad \text{or} \quad \int_0^1 \int_x^1 4x^2 y \, dy \, dx.\]

Computing the first one gives

\[\int_0^1 \int_0^y 4x^2 y \, dx \, dy = \int_0^1 \left( (4/3)x^3 y \right)_0^y \, dy = \int_0^1 (4/3)y^4 \, dy\]
\[ = (4/3)y^{5/5}\bigg|_0^1 = 4/15.\]

The second gives

\[
\int_0^1 \int_x^1 4x^2 y dy dx = \int_0^1 (2x^2 y^2)_{x}^1 dx = \int_0^1 (2x^2 - 2x^4) dx
\]

\[= (2x^3/3 - 2x^5/5)\bigg|_0^1 = 2/3 = 2/5 = (10 - 6)/15 = 4/15.\]