Comparing cycles in positive and mixed characteristic
(Preliminary report)

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The problem we consider is to try to set up a correspondence between cycles over a regular local ring of mixed characteristic with cycles over a regular local ring of positive characteristic.

The motivation comes from Serre's conjectures on intersection multiplicities, which are not completely known in mixed characteristic.

The method is to attempt to apply a construction of Fontaine that goes between rings of positive and of mixed characteristic.
Let $R = V[[X_2, \ldots, X_d]]$, where $V$ is a discrete valuation ring ring with maximal ideal generated by the prime number $p$ and perfect residue field $k$.

$R$ is a regular local ring of mixed characteristic of dimension $d$.

Let $S = k[[T_1, \ldots, T_d]]$.

$S$ is a regular local ring of positive characteristic of dimension $d$.

We would like to set up a correspondence between cycles over $R$ and $S$. We show what we can actually do in this direction.
The Fontaine ring of \( R \)

Let \( R_\infty = \bigcup_n R\left[[p^{1/p^n}, X_1^{1/p^n}, \ldots, X_d^{1/p^n}]\right] \). We then define

\[ E(R_\infty) = \operatorname{lim}_{\leftarrow} R_n, \]

where \( R_n = R_\infty/pR_\infty \) for all \( n \), and the map from \( R_{n+1} \) to \( R_n \) sends \( \bar{r} \) to \( \bar{r}^p \), i.e. it is the Frobenius map.

\( E(R_\infty) \) is the Fontaine ring of \( R_\infty \).

We denote an element of \( E(R_\infty) \) by \( (r_0, r_1, \ldots) \) with \( r_i \in R_\infty \) (taken modulo \( p \)) and \( r_i^p \equiv r_i \) modulo \( p \).

Obvious facts:

1. \( E(R_\infty) \) is a perfect ring of characteristic \( p \).

2. The construction is functorial.
Some not so obvious facts:

1. $E(R_\infty)$ contains $k$.

2. Let

$$T_1 = (p, p^{1/p}, p^{1/p^2}, \ldots),$$

$$T_i = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots) \text{ for } i = 2, \ldots, d.$$ 

Then $E(R_\infty)$ contains the power series ring $k[[T_1, \ldots, T_d]]$, which we denote $S$.

Let $S_\infty$ be the perfect closure of $S$, that is, the ring obtained by adjoining all $p^n$th roots of elements of $S$. Then

$$E(R_\infty) \cong \lim_{\leftarrow} S_\infty / T_1^n S_\infty.$$
Going back from $E(R_\infty)$ to $R_\infty$

We can’t quite reconstruct $R_\infty$ from $E(R_\infty)$, but we can reconstruct the $p$-adic completion of $R_\infty$,

$$\hat{R}_\infty = \lim_{\leftarrow} R_\infty/p^n R_\infty.$$ 

The first step is to take the ring of Witt vectors $W(E(R_\infty))$; since $E(R_\infty)$ is a perfect ring of positive characteristic, this is a reasonable operation.

Then: There is a surjective map

$$W(E(R_\infty)) \to \hat{R}_\infty$$

with kernel generated by $T_1 - p$. Thus the $p$-adic completion of $R_\infty$ can be recovered from $E(R_\infty)$.

We now reformulate the question: is there a correspondence between cycles on $E(R_\infty)$ and cycles on $\hat{R}_\infty$?
The correspondence: to a prime ideal $p$ in $\hat{R}_\infty$ we take the kernel of $E(\hat{R}_\infty)$ to $E(\hat{R}_\infty/p)$ and take the cycle it defines.

Question: does it actually define a cycle, can the kernel be decomposed into a finite sum of prime ideals?

To a prime ideal in $E(R_\infty)$ we take the associated prime ideal in the ring of Witt vectors then divide by $T_1 - p$. Again we have the question of whether we get a finite sum of prime ideals.

The question remains whether this gives an interesting correspondence.
Two simple examples

Let $R = V[[X_2, X_3, X_4]]$

First, take the ideal generated by $X_2 - X_3$. This is a prime ideal in $R$, but not in $R_\infty$ as it is divisible by $X_2^{1/p^n} - X_3^{p^n}$ for all $n$. This corresponds to the ideal in $E(R_\infty)$ generated by $T_2 - T_3$.

Second, take the ideal generated by $X_2 + X_3 + X_4$. This ideal is again clearly prime in $R$, and it is not hard to show that it remains prime in $R_\infty$. It is not clear, however, that it remains prime in $\hat{R}_\infty$; modulo $p$ it of course has $p^n$th roots for all $p$.

If one is really going to study intersection conjectures. For this to give anything one would have to study

$$\hat{R}_\infty/p$$

where $p$ is a minimal prime ideal over the ideal generated by an Eisenstein polynomial in $R$. 