Solutions

You must show all of your work. No work = no credit! Please box your answers and write neatly. I will deduct points for blatantly incorrect notation. Each problem = 9 points.

1. According to the graph of \( f(x) \) above, find the following:

   (a) \( f(-2) = -1 \) - the value of \( f \) at \( x = -2 \) = the closed dot

   (b) Estimate \( f'(-2) = \) slope \( \approx 1 \) or 0.7

   (c) For what \( x \)-value(s) is \( f'(x) = 0 \)?

   \[ \text{horizontal tangent line} \quad \Rightarrow \quad x = -1, \ x = 2 \]

   (d) \( \lim_{x \to -1^-} f(x) = -2 \)

   (e) \( \lim_{x \to 1^+} f(x) = 1 \)

   (f) \( \lim_{x \to 1} f(x) = \text{DNE} \quad \text{since} \quad (d) \neq (e) \)

   (g) \( \lim_{x \to \infty} f(x) = 1 \) - horizontal asymptote

   (h) All \( x \)-value(s) where \( f(x) \) is NOT continuous. State or explain a specific condition from the definition of continuity that fails for each value.

   \[ x = -3 \quad \text{since} \quad \lim_{x \to -3} f(x) = -2 \neq -1 = f(-3) \]

   \[ x = 1 \quad \text{since} \quad \lim_{x \to 1} f(x) \text{ DNE} \]

   \[ x = 3 \quad \text{since} \quad f(3) \text{ DNE} \]
2. (a) Use the definition of the derivative to find \( f'(x) \) for \( f(x) = 2x^2 - 3x \).

\[
\begin{align*}
  f'(x) & = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
  & = \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h} \\
  & = \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) - 3x - 3h - 2x^2 + 3x}{h} \\
  & = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h} \\
  & = \lim_{h \to 0} \frac{4xh + 2h^2 - 3h}{h} \\
  & = \lim_{h \to 0} \frac{h(4x + 2h - 3)}{h} \\
  & = \lim_{h \to 0} 4x + 2h - 3 \\
  & = 4x - 3
\end{align*}
\]

(b) Using the function from part (a) above, find the equation of the tangent line to \( f(x) \) at \( x = 2 \).

\[
\begin{align*}
  \text{slope} & = f'(2) = 4 \cdot 2 - 3 = 5 \\
  \text{point} & : (2, f(2)) = (2, 2) \\
  \text{line} & : y - 2 = 5(x - 2) \quad \text{sufficient} \\
  & y - 2 = 5x - 10 \\
  & y = 5x - 8
\end{align*}
\]
3. Find \( f'(x) \) for each of the following functions. You may leave your answer in any form.

(a) \( f(x) = e^{x(2x+1)^2} \)

\[
\begin{align*}
\quad f'(x) &= e^x \cdot 3 (2x+1)^2 (2) + (2x+1)^3 \cdot e^x \\
&= 6e^x (2x+1)^2 + e^x (2x+1)^3
\end{align*}
\]

(b) \( f(x) = \frac{x^2 - 2x}{x^3 + 1} \)

\[
\begin{align*}
\quad f'(x) &= \frac{(x^3+1) (2x-2) - (x^2-2x)(3x^2)}{(x^3+1)^2}
\end{align*}
\]

(c) \( f(x) = \ln(x^3 - x) \)

\[
\begin{align*}
\quad f'(x) &= \frac{1}{x^3 - x} \cdot (3x^2 - 1)
\end{align*}
\]

(d) \( f(x) = \frac{2}{x} - \sqrt{x} + 2\pi - (3x - 2)^4 + \ln x - e^2 + 3x^\frac{2}{3} + 5 \)

\[
\begin{align*}
\quad f'(x) &= -\frac{2}{x^2} - \frac{1}{3} x^{-\frac{2}{3}} - 4 (3x-2)^3 + \frac{1}{x} + 2 x^{-\frac{1}{3}}
\end{align*}
\]
4. For the function \( f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1 \), find the following:

(a) Find the relative maxima and minima for \( f(x) \). Remember, these are points, not just x-values.

\[
f'(x) = x^2 - x - 2 = 0
\]
\[
(x - 2)(x + 1) = 0
\]
C.V.: \( x = 2 \), \( x = -1 \)

2nd der test: \( f''(x) = 2x - 1 \)

\[
f''(2) = 3 > 0 \Rightarrow \text{C.U.}, \quad f''(-1) = -3 < 0 \Rightarrow \text{C.D.}
\]

(b) Find the intervals on which \( f(x) \) is concave up and concave down.

\[
f''(x) = 2x - 1 = 0
\]
\[x = \frac{1}{2} \text{ is an Important Value}
\]

\[
\begin{array}{c|c|c|c}
 & - & + & \\
T.P. & \frac{1}{2} & & T.P.
\end{array}
\]

Test: \( f''(0) = -1 < 0 \Rightarrow \text{C.D.} \)

\[
f''(1) = 1 > 0 \Rightarrow \text{C.U.}
\]

(c) Find the absolute maximum of \( f(x) \) on the interval \([0, 4]\).

Compare rel. max. to end points

\[
f(-1) = \frac{13}{6} \quad f(0) = 1 \quad f(4) = \frac{19}{3}
\]

\( f(4) = \frac{19}{3} \) is the absolute maximum of \( f \) on \([0, 4]\).
5. Assume that \( y \) is a function of \( x \). Find \( \frac{dy}{dx} \) if \( x^3 - x + y^2 + y = 3 \).

\[
\begin{align*}
\text{take derivative of both sides w.r.t. } x \\
3x^2 - 1 + 2y \frac{dy}{dx} + \frac{dy}{dx} = 0 \\
\text{solving for } \frac{dy}{dx} \\
(2y + 1) \frac{dy}{dx} = 1 - 3x^2 \\
\frac{dy}{dx} = \frac{1 - 3x^2}{2y + 1}
\end{align*}
\]

6. If the marginal cost for a product is \( MC = 5x + 20 \) and the marginal revenue is \( MR = 400 \). The cost of production and sale of 10 units is $1500. What is the total profit function? (Recall that profit is the difference between revenue and cost.)

\[
P = R - C = \int MR \, dx - \int MC \, dx
\]

\[
\Rightarrow P(x) = \int 400 \, dx - \int (5x + 20) \, dx
\]

\[
= 400x - \frac{5}{2}x^2 - 20x + C
\]

\[
P(10) = -1500 \quad \text{(costs are negative)}
\]

plug in: \(-1500 = 400(10) - \frac{5}{2}(10)^2 - 20(10) + C\)

\[
\Rightarrow C = -5050
\]

Rewrite: \( P(x) = 400x - \frac{5}{2}x^2 - 20x - 5050 \)
7. Evaluate the following integrals. Do not forget your constant of integration, if necessary.

(a) \[ \int \frac{6}{3x-1} \, dx \]
\[ u = 3x - 1 \]
\[ du = 3 \, dx \]
\[ 2 \, du = 6 \, dx \]
\[ = \int \frac{2}{u} \, du \]
\[ = 2 \ln |u| + C \]
\[ = 2 \ln |3x-1| + C \]

(b) \[ \int xe^{x^2} \, dx \]
\[ u = x^2 \]
\[ du = 2x \, dx \]
\[ \frac{1}{2} \, du = x \, dx \]
\[ = \int \frac{1}{2} e^u \, du \]
\[ = \frac{1}{2} e^u + C \]
\[ = \frac{1}{2} e^{x^2} + C \]

(c) \[ \int_0^2 (2x^2 - 5x + 1)^3 \, (8x - 10) \, dx \]
\[ u = 2x^2 - 5x + 1 \]
\[ du = (4x - 5) \, dx \]
\[ 2 \, du = (8x - 10) \, dx \]
\[ = \int_0^2 u^{\frac{3}{2}} \cdot 2 \, du \]
\[ = \left[ \frac{u^{\frac{5}{2}}}{4} \cdot 2 \right]_0^2 \]
\[ = \frac{1}{2} \left( 2x^2 - 5x + 1 \right)^4 \bigg|_0^2 \]
\[ = \frac{1}{2} (-1)^4 - \frac{1}{2} (1)^4 \]
\[ = \frac{1}{2} - \frac{1}{2} = 0 \]
8. Find the particular solution to the differential equation \( \frac{dy}{dx} = \frac{x^2 - 4x + 2}{2y + 1} \) subject to the initial condition \( y(0) = 1 \). You may leave your solution in any form you like.

\[
\text{Separate: } (2y + 1) \, dy = (x^2 - 4x + 2) \, dx
\]

\[
\text{Integrate: } \int (2y + 1) \, dy = \int (x^2 - 4x + 2) \, dx
\]

\[
y^2 + y = \frac{1}{3} x^3 - 2x^2 + 2x + C
\]

Plug in \( x = 0 \), \( y = 1 \)

\[
1 + 1 = C \Rightarrow C = 2
\]

Rewrite:

\[
y^2 + y = \frac{1}{3} x^3 - 2x^2 + 2x + 2
\]

9. If the joint cost function is \( C(x, y) = x^3 - 5xy + 5 \), where \( x \) is the labor rate per hour and \( y \) is the cost of raw materials per pound find the following:

(a) \( \frac{\partial C}{\partial x} \) and \( \frac{\partial C}{\partial y} \)

\[
\frac{\partial C}{\partial x} = 3x^2 - 5y
\]

\[
\frac{\partial C}{\partial y} = -5x
\]

(b) If the labor rate remains constant, how does change in the cost of raw material affect the joint cost when the labor rate is $8 per hour and the materials cost $10 per pound?

Evaluate \( \frac{\partial C}{\partial y} \) at \( x = 8 \), \( y = 10 \)

\[
\frac{\partial C}{\partial y} \bigg|_{(8,10)} = -5 \cdot 8 = -40
\]
10. If \( z(x, y) = e^{xy} + y \ln x \), find the following:
\[
\frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y}
\]
\[
\frac{\partial z}{\partial x} = e^{xy} \cdot y + \frac{y}{x}
\]
\[
\frac{\partial z}{\partial y} = e^{xy} \cdot x + \ln x
\]

11. Test for relative maxima and minima of the function
\( z(x, y) = x^2 + 5xy + 10y^2 + 8x - 40y \).
\[
\frac{\partial^2 z}{\partial x^2} = 2x + 5y + 8 \quad \frac{\partial^2 z}{\partial y^2} = 5x + 20y - 40
\]
Set both to 0:
\[
0 \begin{cases} 2x + 5y = -8 \quad \Rightarrow -8x - 20y = 32 \\ 5x + 20y = 40 \quad \Rightarrow 5x + 20y = 40 \\ -3x = 72 \end{cases}
\]
\[
-3x = 72 \quad \Rightarrow x = -24 \quad \text{into (1)} \quad 2(-24) + 5y = -8 \quad \Rightarrow y = 8
\]
Crit. Pt: \((-24, 8, -256)\)

Find D:
\[
\frac{\partial^2 z}{\partial x^2} = 2 \quad \frac{\partial^2 z}{\partial y^2} = 20 \quad \frac{\partial^2 z}{\partial x \partial y} = 5
\]
\[
D = 2 \cdot 20 - 5^2 = 15 > 0 \Rightarrow \text{C.P. is a max/min}
\]

2nd der test:
\[
\left. \frac{\partial^2 z}{\partial x^2} \right|_{(-24, 8, -256)} = 2 > 0 \Rightarrow \text{C.V. \Rightarrow \((-24, 8, -256)\) is a relative minimum}
\]

Good luck in your future studies!

- Emily