1. (Dummit and Foote, §7.1, #26–27) Let $F$ be a field. A discrete valuation on $K$ is defined to be a map $\nu : F^\times \to \mathbb{Z}$ such that

(1): $\nu(ab) = \nu(a) + \nu(b)$;
(2): $\nu$ is surjective; and
(3): $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$

The valuation ring of $\nu$ is defined to be $R = \{x \in F^\times \mid \nu(x) \geq 0\} \cup \{0\}$.

(a) Prove $R$ is a subring of $F$ that contains the identity.
(b) Prove that for each nonzero element $x \in F$, either $x$ or $x^{-1}$ is in $R$.
(c) Prove that an element $x$ is a unit of $R$ if and only if $\nu(x) = 0$.
(d) Now let $F = \mathbb{Q}$ and fix a prime $p$. Define $\nu : \mathbb{Q}^\times \to \mathbb{Z}$ as follows. Fix $x = \frac{a}{b} \in \mathbb{Q}^\times$ write $x = p^n \frac{a'}{b'}$ where $p$ does not divide $a'$ and $b'$. (That is, factor out all possible powers of $p$ and lump them in the $p^n$ term.) Then define

$$\nu(x) = n.$$ 

Describe the corresponding valuation ring $R$.

(e) Retain the setting of part (d). Compute the units in $R$.

2. (Dummit and Foote, §7.2, #5) Let $F$ be a field and consider formal Laurent power series over $F$,

$$F((x)) := \left\{ \sum_{i \geq n} a_i x^n \mid a_i \in F \text{ and } n \in \mathbb{Z} \right\}.$$ 

(a) Define natural operations of addition and multiplication on $F((x))$ and prove that $F((x))$ is a field.
(b) Define $\nu : F((x))^\times \to \mathbb{Z}$ by defining $\nu$ of a formal power series $a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \cdots$ (with $a_n \neq 0$) to be $n$. Prove $\nu$ is a valuation in the sense of Problem 3.
(c) Show that the valuation ring of $\nu$ is the subring of $F((x))$ of formal Laurent series in which no negative powers of $x$ occur. (This subring is called the ring of formal power series in $x$ and typically denoted $F[[x]]$.)

3. (Dummit and Foote, §6.3 #29) Let $R$ be ring. Define the nilradical of $R$ to consider of all elements $x \in R$ such that there exists $n \in \mathbb{Z}$ such that $x^n = 0$.

(a) Suppose $R$ is commutative. Prove that the nilradical of $R$ is an ideal of $R$.
(b) Prove or disprove: if $R$ is arbitrary (not necessarily commutative), then the nilradical of $R$ is an ideal of $R$.

4. (a) Prove or disprove: the subset of $\mathbb{Z}[x]$ consisting of those elements whose coefficients sum to zero is an ideal $\mathbb{Z}[x]$. 

(b) A polynomial \( p \in \mathbb{R}[x] \) is said to vanish to order \( k \) at \( a \in \mathbb{R} \) if \( p^{(j)}(a) = 0 \) for all \( j \leq k \); here \( p^{(j)} \) is the \( j \)th derivative of \( p \). Prove or disprove: for a fixed \( a \in \mathbb{R} \) and \( k \in \mathbb{Z} \), the polynomials that vanish to order \( k \) at \( a \) are an ideal in \( \mathbb{R}[x] \).

(c) Retain the terminology of (b). Prove or disprove: fix \( n \in \mathbb{Z} \), the set polynomials that vanish to order \( n \) at some (not necessarily fixed) point are an ideal in \( \mathbb{R}[x] \).

5. (Dummit and Foote, §7.3 #26) The characteristic of a ring \( R \) is the smallest integer \( n \) such that \( 1 + \cdots + 1 = 0 \); if no such \( n \) exists, \( R \) is said to have characteristic 0.

(a) Prove that the map \( \mathbb{Z} \to R \) defined by

\[
k \mapsto \begin{cases} 
1 + \cdots + 1 & \text{if } k > 0 \\
0 & \text{if } k = 0 \\
-1 - \cdots - 1 & \text{if } k < 0 
\end{cases}
\]

is a ring homomorphism with kernel \( n\mathbb{Z} \) where \( n \) is the characteristic of \( R \).

(b) Suppose \( R \) is commutative with 1 with characteristic \( n > 0 \). Prove that the equality (“The Freshman’s Dream”)

\[
(x + y)^n = x^n + y^n
\]

holds in \( R[x, y] \) if and only if the characteristic \( n \) is prime.

(c) There is a small subtlety in (b). Note that there is a difference between the equality

\[
(1) \quad (x + y)^n = x^n + y^n \quad \text{in } R[x, y]
\]

and the equality

\[
(2) \quad (x + y)^n = x^n + y^n \quad \text{for all } x, y \in R.
\]

Make sure you understand this. Clearly (1) implies (2), but the converse is not clear (and I’m not sure it is even true). In particular for \( R = \mathbb{Z}/n \mathbb{Z} \) one can ask: does (2) hold if and only if \( n \) is prime? This is probably too difficult a question for this problem set. But you should try to think about it nonetheless.

6. Let \( R \) be a (possibly noncommutative) ring. Recall that a proper ideal \( M \) in \( R \) is called a maximal ideal if the only ideals containing it are \( M \) and \( R \). Meanwhile an ideal \( P \) is called prime if whenever \( a \) and \( b \) are elements of \( R \) so that \( ab \in R \), then either \( a \in R \) or \( b \in R \). (When \( R \) is noncommutative, such an ideal is usually called completely prime. But we won’t use this terminology.) Prove or disprove the following implications (each of which hold if \( R \) is commutative):

(i) If \( R/M \) is a field then \( M \) is maximal.
(ii) If \( R/M \) is a skew field then \( M \) is maximal.
(iii) If \( M \) is maximal then \( R/M \) is a field.
(iv) If \( M \) is maximal then \( R/M \) is a skew field.
(v) If \( R/P \) is an integral domain, then \( P \) is prime.
(vi) If $P$ is prime, then $R/P$ is an integral domain.

7. Consider the following four rings

\[ R = \mathbb{Z}[\mathbb{Z}/n] \]
\[ S = \mathbb{Z}[x]/(x^n) \]
\[ T = \mathbb{Z}[x]/(x^n - 1) \]
\[ U = \mathbb{Z}[x]/((x - 1)^n). \]

Determine which of these rings are isomorphic.

8. (Dummit and Foote, §7.5 #3) Let $F$ be a field. Prove that $F$ contains a unique smallest subfield $F_0$ and that $F_0$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p$ for a prime $p$.

9. (Dummit and Foote, §7.5 #4) Use Zorn’s Lemma to prove that the real numbers contain a subring $A$ with $1 \in A$ so that $A$ is maximal (under inclusion) with respect to the property that $\frac{1}{2} \notin A$. Later we will see that $\mathbb{R}$ is the ring of fractions of $A$. 