The Derivative Rules and a Few Examples of Using the Chain Rule

The following Theorems will be used to evaluate each of the derivatives. These Theorems give us simplified ways of evaluating derivatives by not having to apply the definition to each case. For each of the equalities given below, assume that \( f, g \) and other functions of \( x \) have the requisite properties (i.e. differentiability, not equal to zero,...).

\[
\begin{align*}
* \quad & \frac{d}{dx} (c) = 0, \text{ for all } c \in \mathbb{R} \\
* \quad & \frac{d}{dx} (mx + b) = m, \text{ for all } m, b \in \mathbb{R} \\
* \quad & \frac{d}{dx} (x^r) = rx^{r-1} \text{ for all } r \in \mathbb{R}, r \neq 0 \\
* \quad & \frac{d}{dx} ([f(x)]^r) = r [f(x)]^{r-1} \cdot f'(x) \\
* \quad & \frac{d}{dx} (\sin x) = \cos x \\
* \quad & \frac{d}{dx} (\cos x) = -\sin x \\
\uparrow \quad & \frac{d}{dx} (\tan x) = \sec^2 x \\
\uparrow \quad & \frac{d}{dx} (\cot x) = -\csc^2 x \\
\uparrow \quad & \frac{d}{dx} (\sec x) = \sec x \cdot \tan x \\
\uparrow \quad & \frac{d}{dx} (\csc x) = -\csc x \cdot \cot x \\
\end{align*}
\]

Be able to prove the above Theorems that are marked with either a * or a \( \uparrow \). Be able to prove the Theorems with a * will by using the definition of derivative and be able to prove the Theorems with a \( \uparrow \) by using other Theorems (i.e. be able to prove that the derivative of \( \tan x \) is \( \sec^2 x \) given the Quotient Rule and the derivatives of \( \sin x \) and \( \cos x \)).
Here are a few examples of applying these Theorems to finding derivatives. In many cases the final answer has been left unsimplified.

\[
\frac{d}{dx} (4x + 5)^3 = 3(4x + 5)^2(4) = 12(4x + 5)^2
\]

\[
\frac{d}{dx} \left( \left( \frac{2}{x} \right)^4 \right) = \frac{d}{dx} \left( (2x^{-1})^4 \right) = 4 (2x^{-1})^3 (2(-1)x^{-2}) = 4(8x^{-3})(-2x^{-2}) = -64x^{-5}
\]

\[
\frac{d}{dx} (\sin^3(x^2 - 4x + 1)) = 3 (\sin^2(x^2 - 4x + 1)) (\cos(x^2 - 4x + 1))(2x - 4)
\]

\[
\frac{d}{dx} \left( \frac{x^2 + 1}{x^2 - 1} \right) = \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2} = \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}
\]

\[
\frac{d}{dx} \left( \left( \frac{x + 1}{x} \right)^3 \right) = 3 \left( \frac{x + 1}{x} \right)^2 \left( 1 - \frac{1}{x^2} \right)
\]

\[
\frac{d}{dx} (\sin (\cos(x))) = \cos (\cos(x)) (-\sin(x)) = -\sin(x) \cos (\cos(x))
\]