Math 1310 in 100 Easy Steps

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1. This is the summary sheet I handed out at the end of Math 1310-1 at the end of the Spring 2016 semester.
2. Note that the following list is neither complete nor self contained. Each item should trigger your memory of a number of facts, concepts, and logical connections. If it does not consider this a good reason to review the associated material.
3. Prerequisites: know arithmetic, algebra, geometry, cartesian coordinates, functions, the difference between an expression and an equation.
4. Major procedure: come up with a concept, make it precise, derive its properties, and then use the properties to work with the concept.
5. We applied this procedure to three major concepts: limits, derivatives, and integrals.
6. A function \( f \) can be defined by the equation
   \[ y = f(x) = \ldots \]  
   where the dots indicate an appropriate mathematical expression.
7. Functions are important to engineers because they can be used to describe natural phenomena. You want to be able to define a function mathematically from a verbal description.
8. Of course we may use variables other than \( x \) and \( y \).
9. In (1), \( x \) is the independent variable, \( y \) the dependent variable. \( x \) is also called the input, \( y \) the output. The set of all inputs is the domain of \( f \), the set of all outputs the range.
10. The graph of a function is the set of all point \((x, y)\) whose coordinates satisfy the equation \( y = f(x) \).
11. Changing a function corresponds to changes in the graph of the function that often are simple and predictable. For example: Replacing \( x \) by \( x + c \) shifts the graph \( c \) units to the left, replacing \( y \) with \( y + c \) shifts the graph down by \( c \) units, replacing \( x \) with \( cx \) \((c > 0)\) rescales the graph horizontally, replacing \( y \) with \( cy \) rescales it vertically, replacing \( x \) with \(-x\) reflects the graph in the \( y\)-axis, replacing \( y \) with \(-y\) reflects in the \( x\)-axis, and interchanging \( x \) and \( y \) reflects in the line \( y = x \).
12. The output is unique. This corresponds to the vertical line test: A vertical line can intersect the graph of a function in at most one point.
13. A function \( f \) is odd if \( f(x) = -f(-x) \) or even if \( f(x) = f(-x) \) for all \( x \) in its domain. The graph of an even function is symmetric with respect to the \( y\)-axis, the graph of an odd function is symmetric with respect to the origin.
14. Examples of even functions include
   \[ f(x) = x^2, x^4, x^{-2}, \cos x, g(x^2). \]  
   (2)
15. Examples of odd functions include
   \[ f(x) = x^3, x^5, x^{-3}, \sin x, \tan x. \]  
   (3)
16. most functions are neither even nor odd.
17. The zero function is both even and odd.
18. A short catalog of functions:
   - Polynomials: \( f(x) = \sum_{i=0}^{n} a_i x^i \). If \( a_n \neq 0 \) \( n \) is the degree of the polynomial. Low degrees have their own names: constant, linear, quadratic, cubic, quartic, quintic
   - Rational Functions: These can be written as
     \[ R(x) = p(x)/q(x) \]
     where \( p \) and \( q \) are polynomials. It is possible that \( q(x) = 1 \), thus all polynomials are rational functions.
− **Exponential Functions**: \( f(x) = Aa^x, \ a > 0, \ a \neq 1 \). The natural exponential has base \( e \),
\[
\exp(x) = e^x
\] (5)
where
\[
e = 2.71828182845904523536028747135\ldots
\] (6)

− **Logarithms** are defined by \( a^{\log_a x} = x \) or \( \log_a a^x = x \). Thus logarithms are the inverse of exponentials. The natural logarithm has base \( e \),
\[
\ln x = \log_e x.
\] (7)

− **Trigonometric Functions**: \( f(x) = \sin x, \ \cos x, \ \tan x \). Angles are measured in radians, the length of the arc on the unit circle. \( \cos x \) and \( \sin x \) are the coordinates of a point on the unit circle corresponding to the unit circle.

19. Functions can be combined to give new functions. Let \( f \) and \( g \) be two functions, with appropriate domains and ranges (what does “appropriate” mean in this context?). Then
\[
(f + g)(x) = f(x) + g(x) \\
(f - g)(x) = f(x) - g(x) \\
(f \times g)(x) = f(x) \times g(x) \\
(f \div g)(x) = f(x) \div g(x) \\
(f \circ g)(x) = f(g(x))
\] (8)

20. Function composition does not commute, in general
\[
f \circ g \neq g \circ f.
\] (9)

21. The inverse \( f^{-1} \) of a function \( f \) is defined by the equivalent equations
\[
f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.
\] (10)

22. \( f^{-1} \) is pronounced \( f\)-inverse and does not equal the reciprocal of \( f \).
\[
f^{-1}(x) \neq \frac{1}{f(x)}.
\] (11)

23. To find the inverse solve the equation \( y = f(x) \) for \( x \) and then interchange \( x \) and \( y \).

24. The graph of \( f^{-1} \) is the graph of \( f \) reflected in the line \( y = x \).

25. A function may not have an inverse. It has an inverse if and only if it satisfies the horizontal line test.

26. Logarithms are the inverses of exponentials, and vice versa.

27. **Properties of exponentials** include
\[
a^x a^y = a^{x+y}, \quad \frac{a^x}{a^y} = a^{x-y}, \quad (a^x)^y = a^{xy}, \quad \text{quad} (a^x b^y) = (ab)^{x+y}, \quad a^0 = 1.
\] (12)

28. **Properties of logarithms** include
\[
\log(uv) = \log u + \log v, \quad \log \frac{u}{v} = \log u - \log v, \quad \log u^v = v \log u
\] (13)

29. All logarithms are proportional:
\[
\log_b x = \frac{\log_a x}{\log_a b}
\]  

(14)

30. The essence of exponential growth is that over any interval of fixed length the function value is multiplied with a fixed factor. Another way of putting this is that over intervals of given length the exponential increases by a constant percentage.

31. You want to be able to define exponentials with given doubling or tripling times, and to express such exponentials, for examples as

\[ f(t) = A2^{t/D} = Ae^{kt} . \]  

(15)

32. You should be able to solve simple logarithmic or exponential equations.

33. Examples for exponential growth include compound interest and population growth at a constant percentage.

34. A parametric curve is a curve that can be described by

\[ x = f(t), \quad y = g(t), \quad a \leq x \leq b. \]  

(16)

35. (Definition, page 95). We write

\[ \lim_{x \to a} f(x) = L \]  

and say “the limit of \( f(x) \), as \( x \) approaches \( a \), equals \( L \)”, if we can make the values of \( f(x) \) arbitrarily close (as close to \( L \) as we like) by taking \( x \) to be sufficiently close to \( a \) (on either side of \( a \)) but not equal to \( a \).

36. Our original motivation for studying limits was our need to understand velocities and tangent lines.

37. Suppose \( s(t) \) gives the distance we have traveled (e.g., along a highway) at time \( t \). We refer to \( s \) as location.

38. Then the average velocity in the interval \([a, a + h]\) is

\[ \text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{s(a + h) - s(a)}{h} \]  

(18)

39. If we think of this in terms of just the graph of \( s \) (without reference to the physics) this becomes

\[ \text{slope of secant line} = \frac{\text{rise}}{\text{run}} = \frac{s(a + h) - s(a)}{h} \]  

(19)

40. The technique we used most frequently to compute limits consisted of manipulating the expression defining \( f(x) \) so as to turn it into an equivalent expression that equals \( f(x) \) for all \( x \neq a \) but that can be evaluated at \( x = a \).

41. We extended our concepts to one-sided limits and to limits involving infinity. You want to be able to make sense of the following notations:

\[ \lim_{x \to a^-} f(x), \quad \lim_{x \to a^+} f(x), \quad \lim_{x \to \infty} f(x), \quad \lim_{x \to -\infty} f(x), \quad \lim_{x \to a} f(x) = \infty \]  

(20)

42. Here are some typical examples. You want to work through these if you don’t remember how to do them.
\[
\lim_{x \to -n} |x| =
\lim_{x \to 0} x^2 \cos x^2 =
\lim_{x \to \infty} \sqrt{x^2 + x} - x =
\]

43. More generally, look for the examples in the notes of January 27 to February 8 and make sure you understand how we worked those examples.

44. Limit Laws (p. 104): Suppose that \( c \) is a constant and the limits \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist. Then

\[
\begin{align*}
L[f(x) + g(x)] &= Lf(x) + Lg(x) \\
L[f(x) - g(x)] &= Lf(x) - Lg(x) \\
L[cf(x)] &= cLf(x) \\
L[f(x) \times g(x)] &= Lf(x) \times Lg(x) \\
L \frac{f(x)}{g(x)} &= \frac{Lf(x)}{Lg(x)}, \quad \text{if} \quad Lg(x) \neq 0
\end{align*}
\]

(21)

45. The Squeeze Theorem If

\[
f(x) \leq g(x) \leq h(x)
\]

for \( x \) near \( a \) (except possibly at \( x = a \)) and

\[
Lf(x) = Lh(x) = L
\]

then then

\[
Lg(x) = L.
\]

(24)

46. A function \( f \) is **continuous at a number** \( a \) if

\[
f(a) = \lim_{x \to a} f(x).
\]

(25)

47. \( f \) is continuous if it is continuous at all points in its domain.

48. Note that (25) says 3 things:
1. \( f(a) \) exists, i.e., \( a \) is in the domain of \( f \).
2. \( Lf(x) \) exists.
3. \( Lf(x) = f(a) \)

49. **Intermediate Value Theorem** (page 120) Suppose that \( f \) is continuous on the closed interval \([a, b]\), and let \( N \) be any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \). Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \).

50. Bisection Method

51. Know how to identify horizontal and vertical asymptotes.

52. A few of the questions have answers that are polynomials or rational expressions, and ask you to give the answer in standard form. The standard form of a polynomial is

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.
\]

(26)

The standard form of a rational expression is \( \frac{N}{D} \) where \( N \) and \( D \) are polynomials in standard form.

53. Here are some definitions and notations for a derivative:

\[
f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = D_x f(x).
\]

(27)
54. **Onion Method.** Apply the rule that is appropriate for the last operation needed to evaluate the expression. Repeat for the ingredients of that expression.

Examples:

\[
f(x) = \frac{\sin x^2}{\sin e^x} =
\]

\[
f(x) = \frac{e^{x^2 + \sin x}}{\cos x^2} =
\]

(28)

55. You want to understand derivatives algebraically and geometrically.

56. In order actually to compute derivatives we usually use **Differentiation Rules**, see Table 1.

57. **Implicit Differentiation.** Differentiate in an equation. Be clear with respect to which variable you differentiate, and which variables depend on it.

58. Example, assuming \( y = y(x) \).

\[
x^2 + y^2 = 1 \implies 2x + 2yy' = 0 \implies y' = -\frac{x}{y}.
\]  

(29)

59. One application of implicit differentiation is the computation of the derivative of an inverse function. For example, the natural logarithm is the inverse of the exponential, and the derivative of the exponential is the exponential itself. Starting with

\[
e^{\ln x} = x
\]  

(30)

we obtain by differentiating (and using the chain rule)

\[
e^{\ln x} \frac{d}{dx} \ln x = 1
\]  

(31)

from which we get

\[
\frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}.
\]  

(32)

60. **Logarithmic Differentiation.** Take a logarithm first, differentiate implicitly, solve for the derivative. Example:
\[ y = x^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \ln x \]
\[ \Rightarrow \quad \frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x} \]
\[ \Rightarrow \quad y' = x^{\sin x} \left( \cos x \ln x + \frac{\sin x}{x} \right). \]  

(33)

61. Of course, we don’t have to use \( x \) and \( y \) as variables. We could use any others.

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### Differentiation Rules

\[
\begin{align*}
\frac{d}{dx} x^r &= r x^{r-1} & \text{Power Rule} \\
\frac{d}{dx} \sin x &= \cos x & \text{sin} \\
\frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1 - x^2}} & \text{arcsin} \\
\frac{d}{dx} \cos x &= -\sin x & \text{cos} \\
\frac{d}{dx} \arccos x &= \frac{-1}{\sqrt{1 - x^2}} & \text{arccos} \\
\frac{d}{dx} \tan x &= \frac{1}{\cos^2 x} & \text{tan} \\
\frac{d}{dx} \arctan x &= \frac{1}{1 + x^2} & \text{arctan} \\
\frac{d}{dx} e^x &= e^x & \text{exponential} \\
\frac{d}{dx} \ln x &= \frac{1}{x} & \text{logarithm} \\
\frac{d}{dx} \sinh x &= \cosh x & \text{hyperbolic sin} \\
\frac{d}{dx} \cosh x &= \sinh x & \text{hyperbolic cos} \\
\frac{d}{dx} \tanh x &= \frac{1}{\cosh^2 x} & \text{hyperbolic tan} \\
(cf)' &= cf' & \text{Constant Factor Rule} \\
(f \pm g)' &= f' \pm g' & \text{Sum and Difference Rules} \\
(fg)' &= f'g + fg' & \text{Product Rule} \\
\left( \frac{f}{g} \right)' &= \frac{f'g - fg'}{g^2} & \text{Quotient Rule} \\
\frac{d}{dx} f(g(x)) &= f'(g(x))g'(x) & \text{Chain Rule}
\end{align*}
\]

Table 1: Differentiation Rules

62. You want to understand what we mean by a local, global, absolute, or relative, minimum, maximum, or extreme, value of a function.

63. The function that you are minimizing or maximizing is sometimes called the objective function.
64. Extreme values can occur only at critical numbers or endpoints of an interval. Critical numbers are values of the independent variable. There are two kinds:
   - singular points where the derivative does not exist,
   - stationary points where the derivative is zero.
65. Thus a standard procedure for finding extreme values involves identifying evaluating the objective function at all critical values and all endpoints of intervals and examining what happens there.
66. Of the three kinds of values, stationary points occur most frequently and are the most important.
67. The function value at a stationary point is
   - a local minimum if the first derivative changes sign from negative to positive,
   - a local minimum if the second derivative is positive,
   - a local maximum if the first derivative changes sign from positive to negative,
   - a local maximum if the second derivative is negative.
68. When solving word problems, keep in mind which variables denote constants, which variable denotes the function you want to minimize or maximize, and what is the independent variable. In many word problems, the objective function depends on several variables that are interrelated. In that case you first need to choose one variable as the independent variable and express the others in terms of your chosen independent variable.
69. The above criteria are easy to understand and remember if you understand the relationship between derivatives and the shape of a graph.
70. A function is
   - increasing if the first derivative is positive,
   - decreasing if the first derivative is negative,
   - concave up (also called convex) if the second derivative is positive,
   - concave down (also called concave) if the second derivative is negative.
71. The graph of a function has a horizontal tangent at a point where the derivative is zero (naturally).
72. A point of inflection, or an inflection point, is a point on the graph where the graph changes from being concave up to being concave down, or vice versa.
73. We have a point of inflection if the second derivative changes sign.
74. You want to be able to use these facts to draw the graph of a function.
75. You also want to be able to check that these facts are consistent with the graph of a specific function.
76. To solve \( f(x) = 0 \) we can use Newton’s Method:

\[
x_0 \text{ given, } \quad x_{k+1} = g(x_k) \quad \text{where} \quad g(x) = x - \frac{f(x)}{f'(x)}.
\] (35)

This is a special case of a fixed point iteration.
77. The phrase Related Rates refers to derivatives of time dependent functions that are related. Typical ingredients of related rates problems are: time does not occur explicitly, you differentiate implicitly, and you differentiate before you evaluate!
78. A function \( F \) is an antiderivative of a function \( f \) if \( F' = f \). Antiderivatives are determined up to a constant.
79. An indefinite integral is an expression of the form

\[
\int f(x) \, dx.
\] (36)

It either denotes the set of all antiderivatives of \( f \), or a particular one, depending on the context. \( \int \) is the integration symbol, \( f \) is the integrand, and \( x \) is the integration variable.
80. A definite integral

\[
\int_a^b f(x) \, dx
\] (37)
is a number. \(a\) and \(b\) are the lower and upper limits of integration respectively. If \(f\) is non-negative in the interval \([a, b]\) then \(\int_a^b f(x)\,dx\) is the area of the region enclosed by the curves \(y = 0, x = a, x = b,\) and \(y = f(x)\).

81. We discussed alternative interpretations of definite integrals. If \(x\) is time and \(f(x)\) is acceleration then the definite integral is velocity, if \(f\) is velocity then the definite integral is location.

82. We defined definite integrals as the limit of a Riemann Sum: Let \(\Delta x = \frac{b-a}{n}\) and \(x_i = a + i\Delta x, \ i = 0, \ldots, n.\) (38)

Then

\[
\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{39}
\]

where the \(x_i^*\) are arbitrary points in the interval \([x_{i-1}, x_i]\). The important thing about Riemann sums is to read them backwards: recognize a Riemann sum as an integral.

83. The **Fundamental Theorem of Calculus** makes two equivalent statements:

\[
\frac{d}{dx} \int_a^x f(t)\,dt = f(x) \quad \text{and} \quad \int_a^b f(x)\,dx = F(b) - F(a) \tag{40}
\]

where \(F\) is any antiderivative of \(f\).

84. Thus a definite integral can be evaluated by finding an antiderivatives of the integrand, evaluating it at the limits of integration, and computing the difference.

85. Every differentiation rule is also an integration rule, just read it in the opposite direction.

86. **Integration by substitution** is the inverse process of the chain rule:

\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \quad \iff \quad \int f'(g(x))g'(x)\,dx = f(g(x)) \tag{41}
\]

87. We often formalize substitution by writing

\[
u = g(x) \quad \text{and} \quad du = g'(x)\,dx\]

(42)

With this substitution we get

\[
\int f'(g(x))g'(x)\,dx = \inf f(u)\,du. \tag{43}
\]

88. When using substitution to compute an indefinite integral we return to the original variable.

89. For definite integrals this is not necessary, but the limits of integration need to be replaced by their corresponding \(u \) values.

90. **Integration by parts:**

\[
\int uv' = uv - \int u'v \quad \text{and} \quad \int_a^b uv' = uv \bigg|_a^b - \int_a^b u'v \tag{44}
\]

is the inverse process of the product rule.

91. Proper rational expressions can be integrated if we can factor the denominator. For example

\[
\int \frac{L(x)}{(x-a)(x-b)}\,dx = \int \frac{A}{x-a}\,dx + \frac{B}{x-b}\,dx \tag{45}
\]

where \(L(x)\) is any linear function and \(A\) and \(B\) are constants that can be computed by recombining the terms on the right and then matching the numerators.
92. You always want to be on the lookout for **symmetry** in a problem and make use of it when you can. For example:

\[ \int_{-b}^{b} f(x) \, dx = \begin{cases} 2 \int_{0}^{b} f(x) \, dx & \text{if } f \text{ is even} \\ 0 & \text{if } f \text{ is odd} \end{cases} \]  

(46)

93. **Integration Rules** include, for example,

\[ \int k f(x) \, dx = k \int f(x) \, dx, \]

\[ \int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx \]

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]  

(47)

The first two properties mean that integration is **linear**, just like differentiation is linear.

94. A definite integral is **improper** if there are one or two infinite limits of integration, or the integrand has one or more infinite discontinuities. In all cases we define and compute the integral by taking one or more appropriate limits. If the limits exist we say that the integral **converges**, if they do not we say that the integral **diverges**. We’ll illustrate the ideas with some examples, and then give specific definitions.

\[ \int_{-\infty}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_{a}^{b} f(x) \, dx, \]

\[ \int_{-\infty}^{b} f(x) \, dx = \lim_{a \to -\infty} \int_{a}^{b} f(x) \, dx \]

\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{\infty} f(x) \, dx \]  

(48)

95. In the third of these formulas, any real number \( a \) can be used.

96. If the integrand is discontinuous at \( a \) we define

\[ \int_{a}^{b} f(x) \, dx = \lim_{t \to a^+} \int_{t}^{b} f(x) \, dx. \]  

(49)

97. If the integrand is discontinuous at \( b \) we define

\[ \int_{a}^{b} f(x) \, dx = \lim_{t \to b^-} \int_{a}^{t} f(x) \, dx. \]  

(50)

98. If the integrand is discontinuous at \( c \) where \( a < c < b \) we define

\[ \int_{a}^{b} f(x) \, dx = \lim_{t \to c^-} \int_{a}^{t} f(x) \, dx + \lim_{t \to c^+} \int_{t}^{b} f(x) \, dx. \]  

(51)

99. If the integrand has several discontinuities we break the interval into suitable subintervals.

100. In all cases, if all limits involved exist we say that the integral **converges**, if they do not we say it **diverges**.