1. Find the Fourier sine series of the function

\[ f(x) = \begin{cases} 0, & x = 0, \pi \\ 100, & 0 < x < \pi \end{cases} \]

extended to be periodic of period \( 2\pi \).

2. Suppose \( u(x,t) \) gives the temperature at a point \( x \) units from the left endpoint of a wire of length \( \pi \) at time \( t \). Then \( u \) satisfies the heat equation: \( u_t(x,t) = c^2 u_{xx}(x,t) \) (\( c \) is some constant based on the wire). Suppose now that the ends of the wire are stuck in ice: then the boundary conditions \( u(0,t) = u(\pi,t) = 0 \) are satisfied (at least until the ice melts!). Suppose further that there is an initial temperature distribution in the wire \( u(x,0) = f(x) \).

a. Suppose \( u(x,t) = X(x)T(t) \). Explain how to obtain the equations:

\[
\begin{align*}
X'' - kX &= 0 \\
T' - kc^2T &= 0
\end{align*}
\]

for some constant \( k \).

b. Suppose the wire is not a constant temperature of 0 degrees (i.e., \( f(x) \neq 0 \)). Explain why \( k \) must be negative, so the equations may be rewritten:

\[
\begin{align*}
X'' + \mu^2 X &= 0 \\
T' + \mu^2 c^2 T &= 0
\end{align*}
\]

where \( \mu = -\sqrt{k} \).

c. Show that \( X(x) = a \sin \mu x \), and that \( \mu \) must be an integer. Set \( X_n(x) = \sin nx \).

d. Solve \( T_n''(t) + (nc)^2 T_n(t) = 0 \) to find \( T_n(t) \). (Answer: \( T_n(t) = b_n e^{-nc^2 t} \) for some constant \( b_n \).

e. Write a general formula for \( u \), and use the initial condition to show that

\[
b_n = 2 \pi \int_0^\pi f(x) \sin(nx) dx
\]

f. Suppose the wire is heated uniformly to 100 degrees, then the ends are stuck in ice. Assume \( c = 1 \) for the wire. Find \( u(x,t) \).

3. Show that \( u(x,t) = -\kappa u_0(x,t) + k(x,t) \) (where \( \kappa \) is a given constant and \( k(x,t) \) is a given function) is solved by \( u(x,t) = e^{-\kappa t} f(x-\kappa t) \), where \( f \) is any smooth function.

4. Solve the equation \( x^2 u_{xx} + y^2 u_y = 0 \) (your solution will involve an arbitrary function \( f \)).

5. Solve the homogeneous wave equation for a string with length 1 and \( c = 1 \) with initial conditions \( u(x,0) = \sin \pi x \), \( u_t(x,0) = 0 \).