Field Patterns: A new type of Wave

Ornella Mattei and Graeme Milton,
Department of Mathematics, The University of Utah


설계 시
상담
32
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Space&Time
Architecture

Formulation of the problem

- Generic wave equation:

$$\frac{\partial}{\partial x}\left(\alpha(x,t)\frac{\partial u(x,t)}{\partial x}\right)-\frac{\partial}{\partial t}\left(\beta(x,t)\frac{\partial u(x,t)}{\partial t}\right)=0$$

The coefficients are time – dependent $\rightarrow$ DYNAMIC MATERIALS

- Boundary conditions: The medium is infinite in the x–direction

- Initial conditions:

$$u(x,0)=g(x)$$
$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}=f(x)$$

[see, e.g., Lurie, 2007]

Realization of dynamic materials

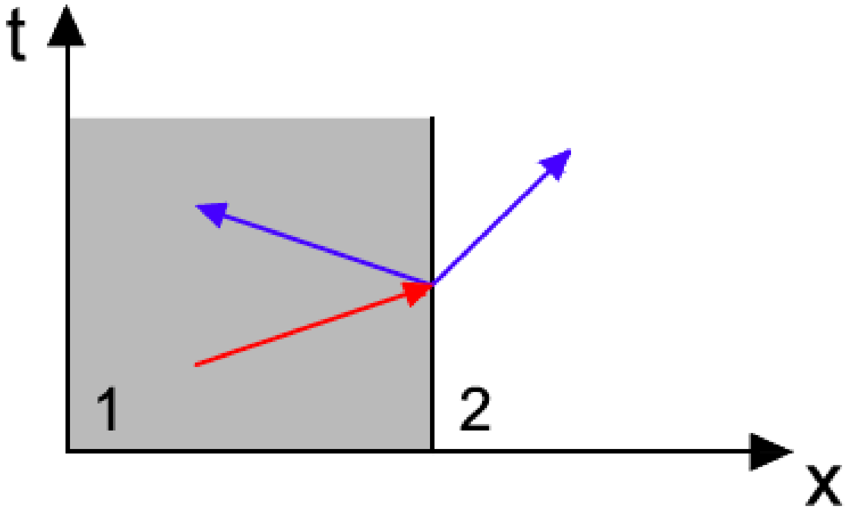
- Liquid crystals
- Ferroelectric, ferromagnetic materials
- Pump wave + small amplitudes waves [e.g. Louisell & Quate (1958)]
- Transmission line with modulated inductance [e.g. Cullen (1958)]
- Experiments and more references in [Honey & Jones (1958)]
- Dynamic modulation + photons [e.g. Fang et al. (2012), Boada et al. (2012), Celi et al. (2014), Yuan et al. (2016)]

...

- Walking droplets [e.g. Couder et al. (2005), Couder & Fort (2006), Bush (2015)]
- Time reversal [e.g. Fink (2016), Goussev et al. (2016)]

Dynamic composites

Pure space interface

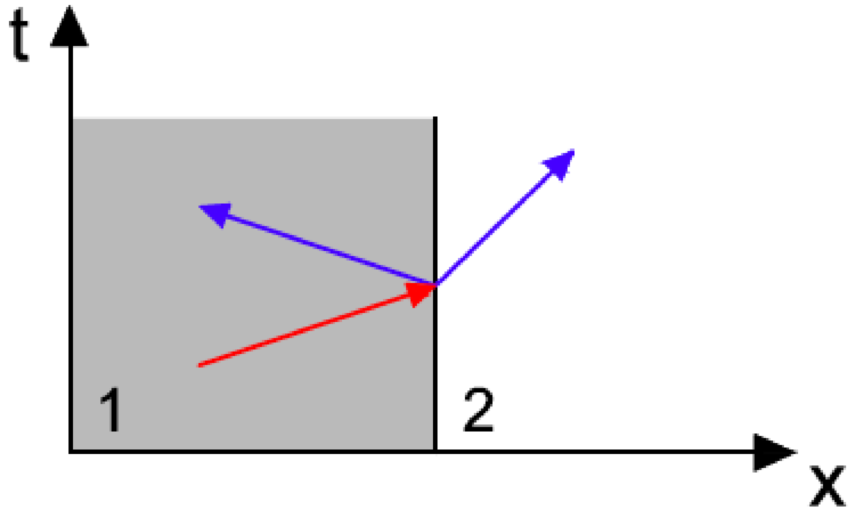


Pure time interface

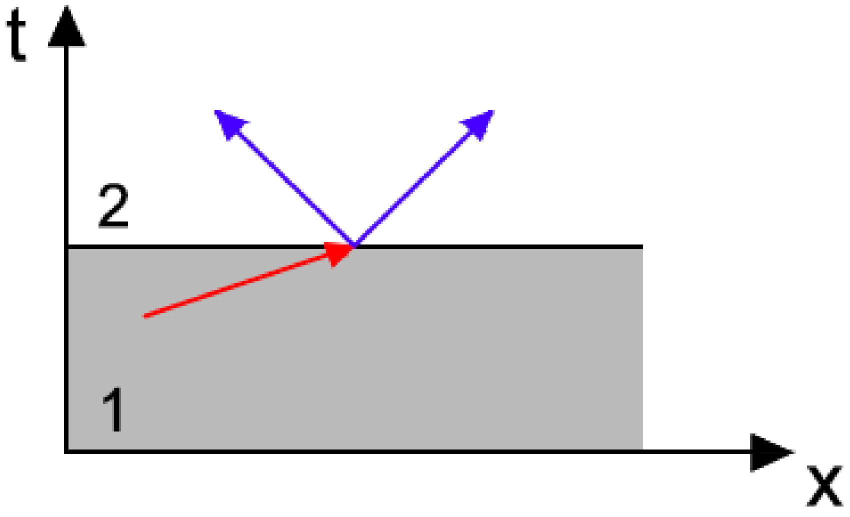
?

Dynamic composites

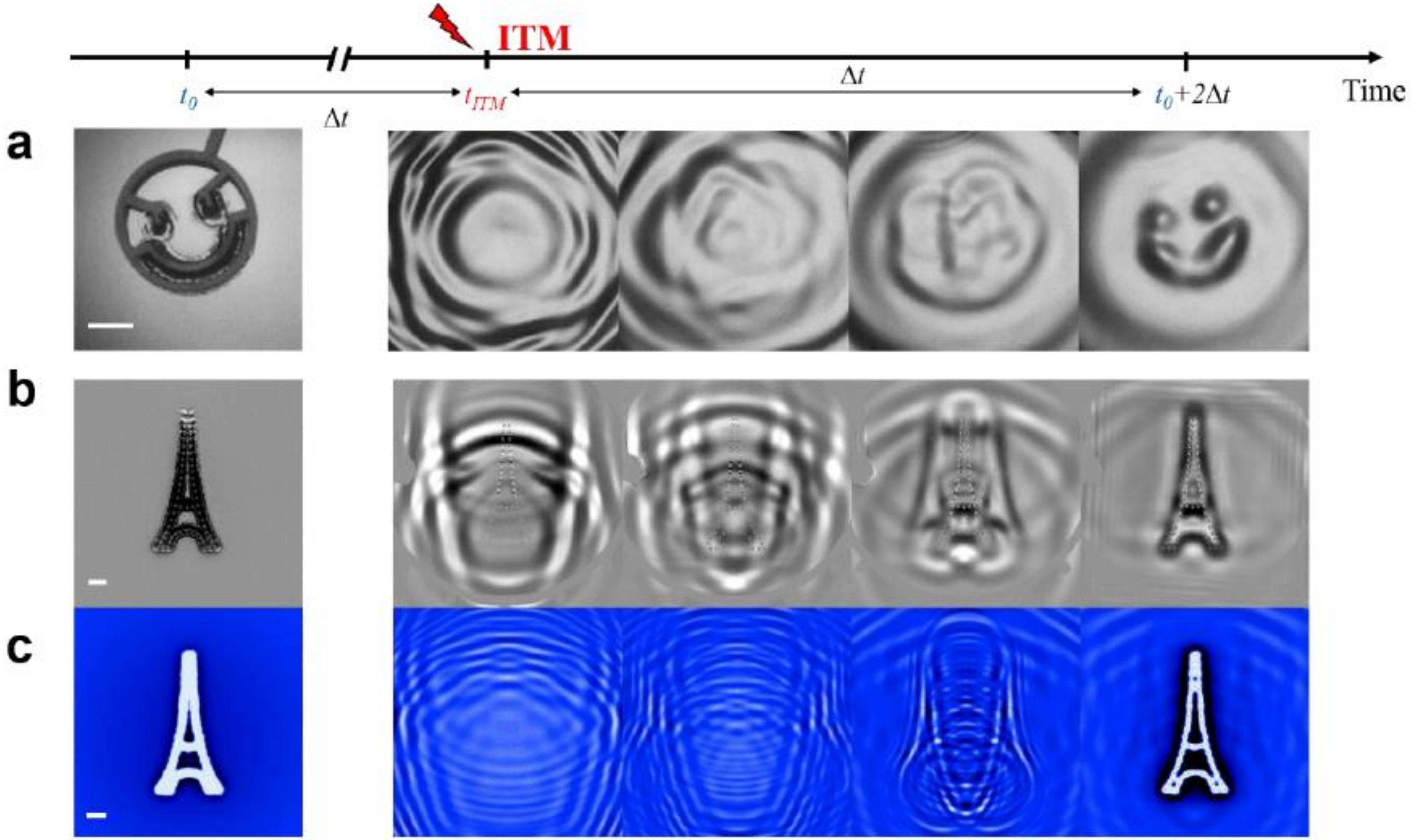
Pure space interface



Pure time interface



What happens at a time interface?



Bacot, Labousse, Eddi, Fink, and Fort, Nature 2016

Thinking of the wave equation as a conductivity problem

$$\mathbf{j}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \text{where} \quad \nabla\cdot\mathbf{j} = 0, \quad \mathbf{e} = -\nabla V,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{pmatrix} \alpha(\mathbf{x}) & 0 \\ 0 & -\beta(\mathbf{x}) \end{pmatrix}, \quad \begin{array}{lcl} \text{material } 1 & \rightarrow & \alpha_1, \beta_1 \\ \text{material } 2 & \rightarrow & \alpha_2, \beta_2 \end{array}$$

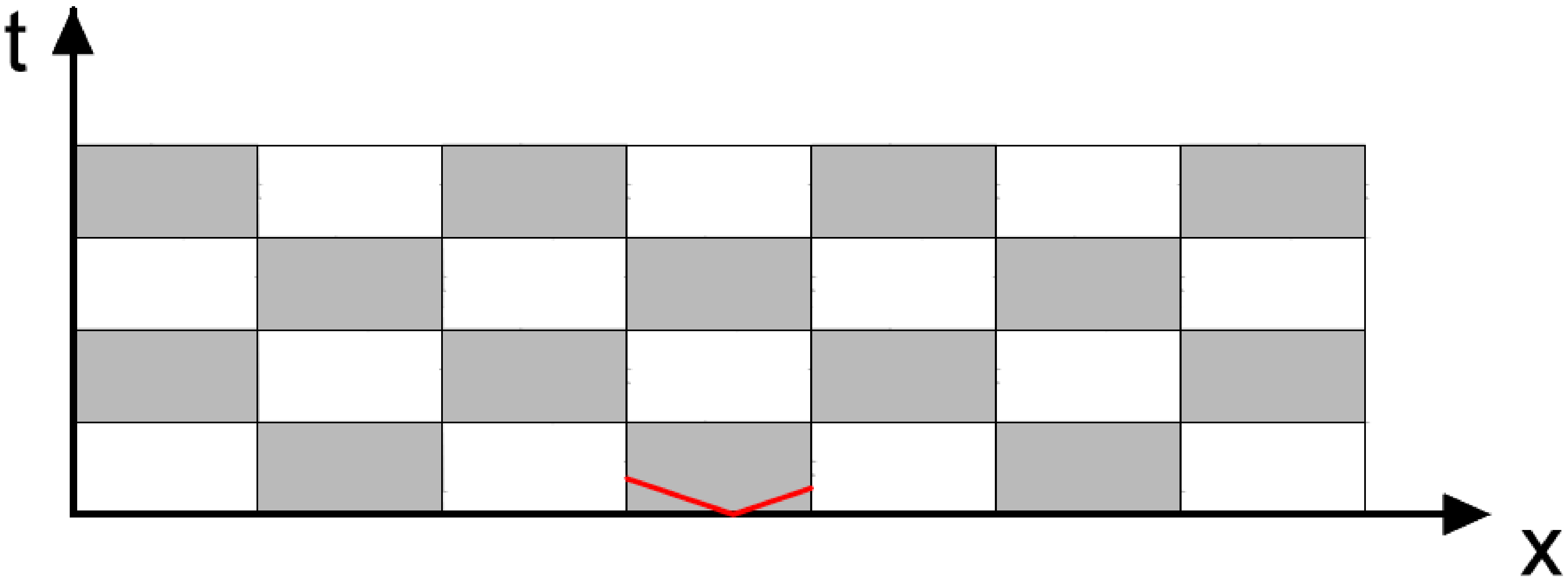
$$\boxed{\frac{\partial}{\partial x_1}\left(\alpha(x_1,x_2)\frac{\partial V(x_1,x_2)}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\beta(x_1,x_2)\frac{\partial V(x_1,x_2)}{\partial x_2}\right) = 0}$$

N.B. Hyperbolic materials!! [See, e.g. the review Poddubny, Iorsh, Belov, Kivshar, 2013]

$$\boxed{\alpha_i\frac{\partial^2 V_i}{\partial x^2} - \beta_i\frac{\partial^2 V_i}{\partial t^2} = 0, \qquad i = 1,2}$$

D'Alembert solution : $V_i(x,t) = V_i^+(x - c_i t) + V_i^-(x + c_i t) \qquad c_i = \sqrt{\dfrac{\alpha_i}{\beta_i}}$

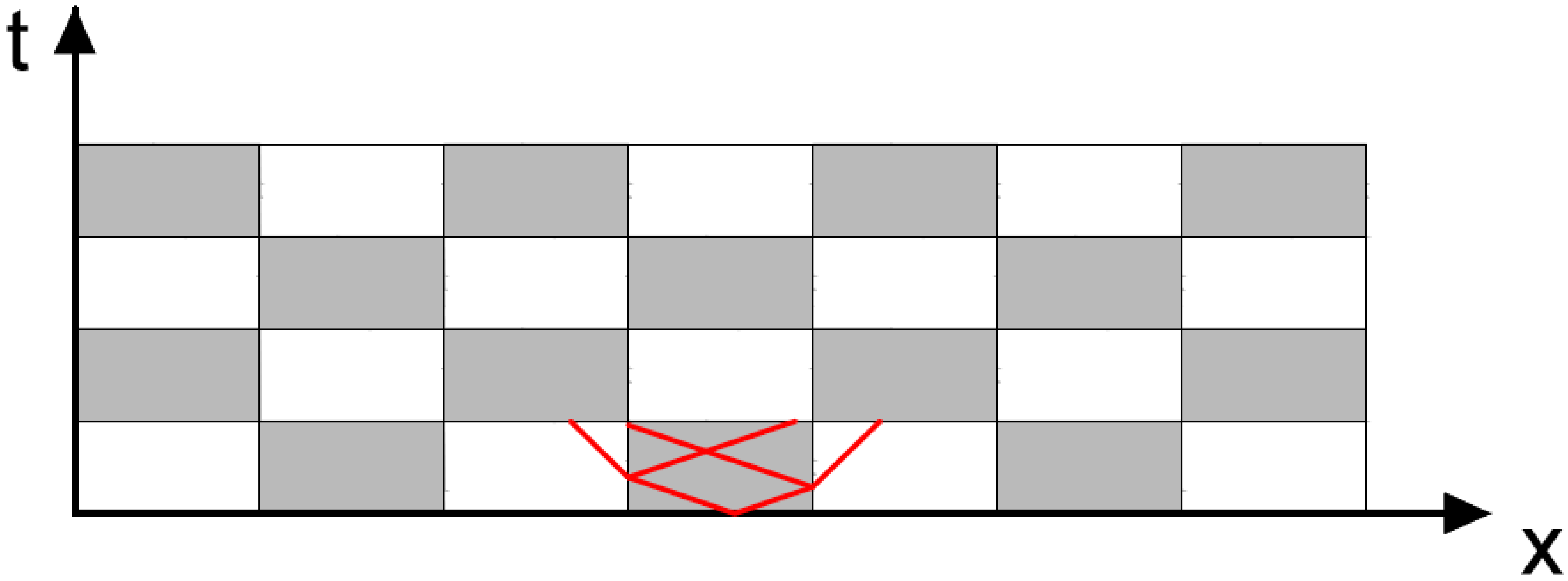
Evolution of a disturbance in a space-time checkerboard



Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \boldsymbol{\sigma}_1 \nabla V_1 = \mathbf{n} \cdot \boldsymbol{\sigma}_2 \nabla V_2 \end{cases}$$

Evolution of a disturbance in a space-time checkerboard



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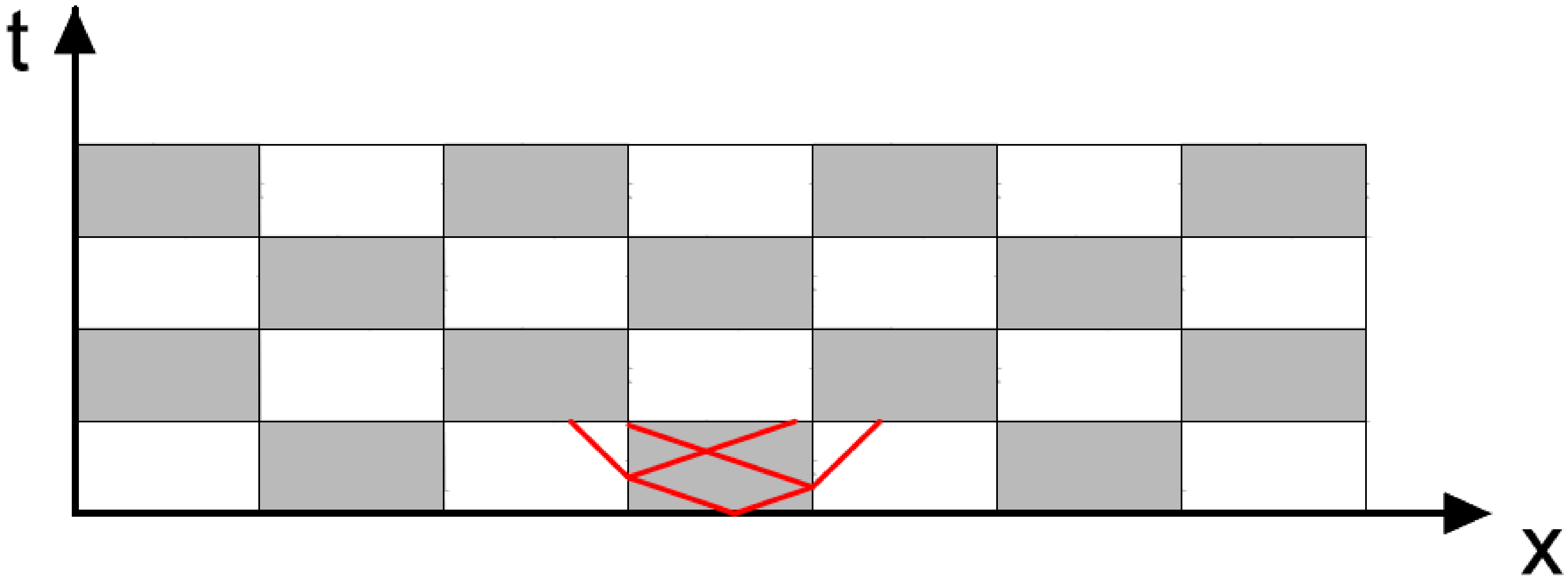

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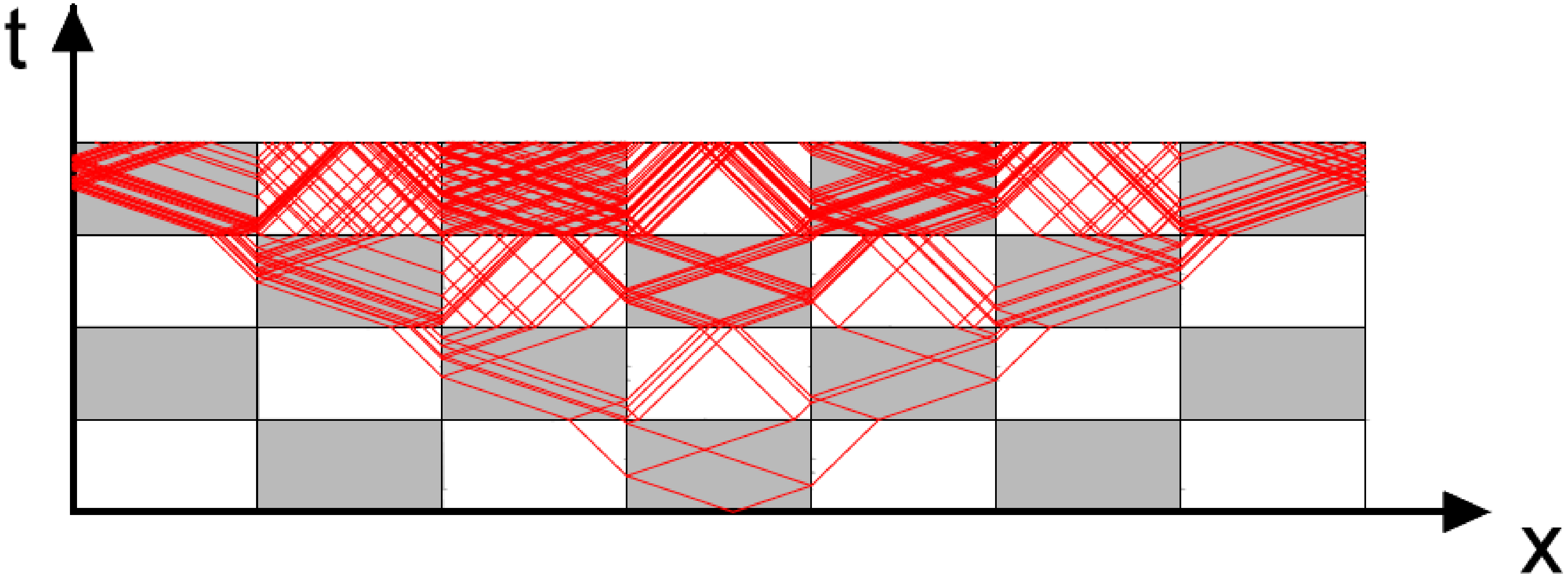
Evolution of a disturbance in a space-time checkerboard

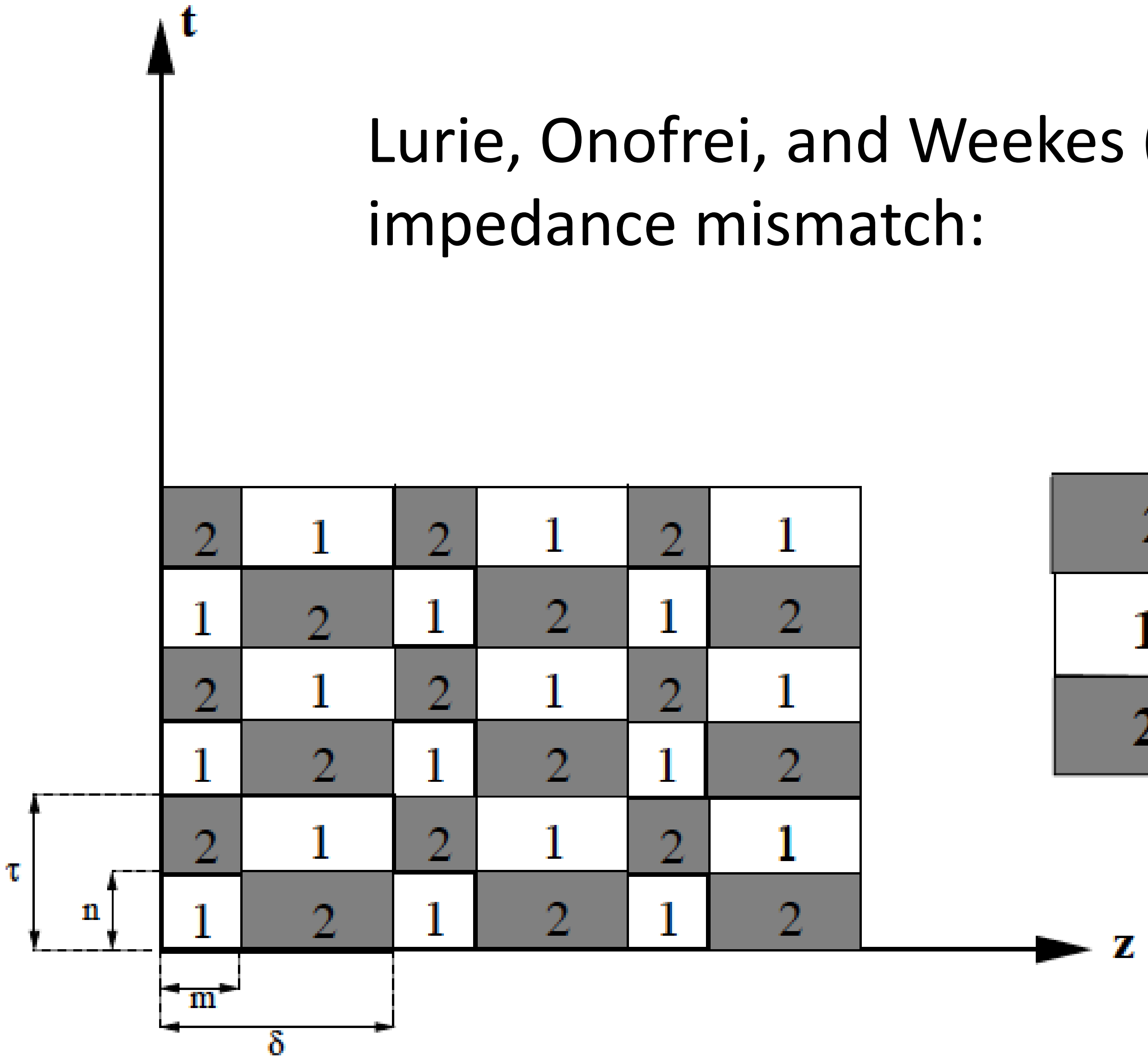


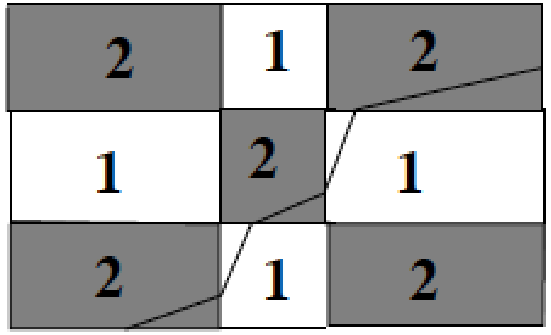
Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \boldsymbol{\sigma}_1 \nabla V_1 = \mathbf{n} \cdot \boldsymbol{\sigma}_2 \nabla V_2 \end{cases}$$

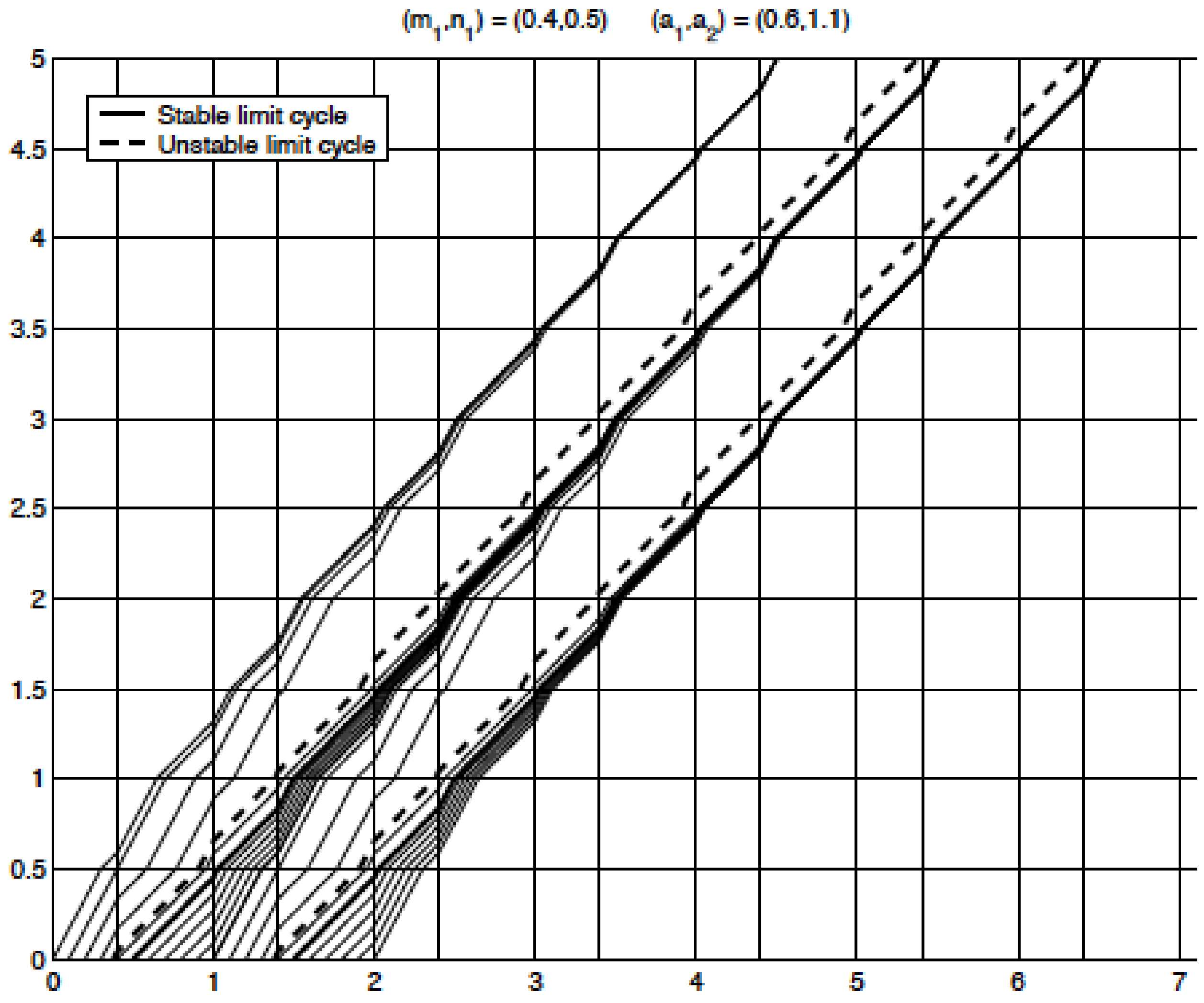
How to avoid this complicated cascade?

Lurie, Onofrei, and Weekes (2009) suggested having a zero impedance mismatch:





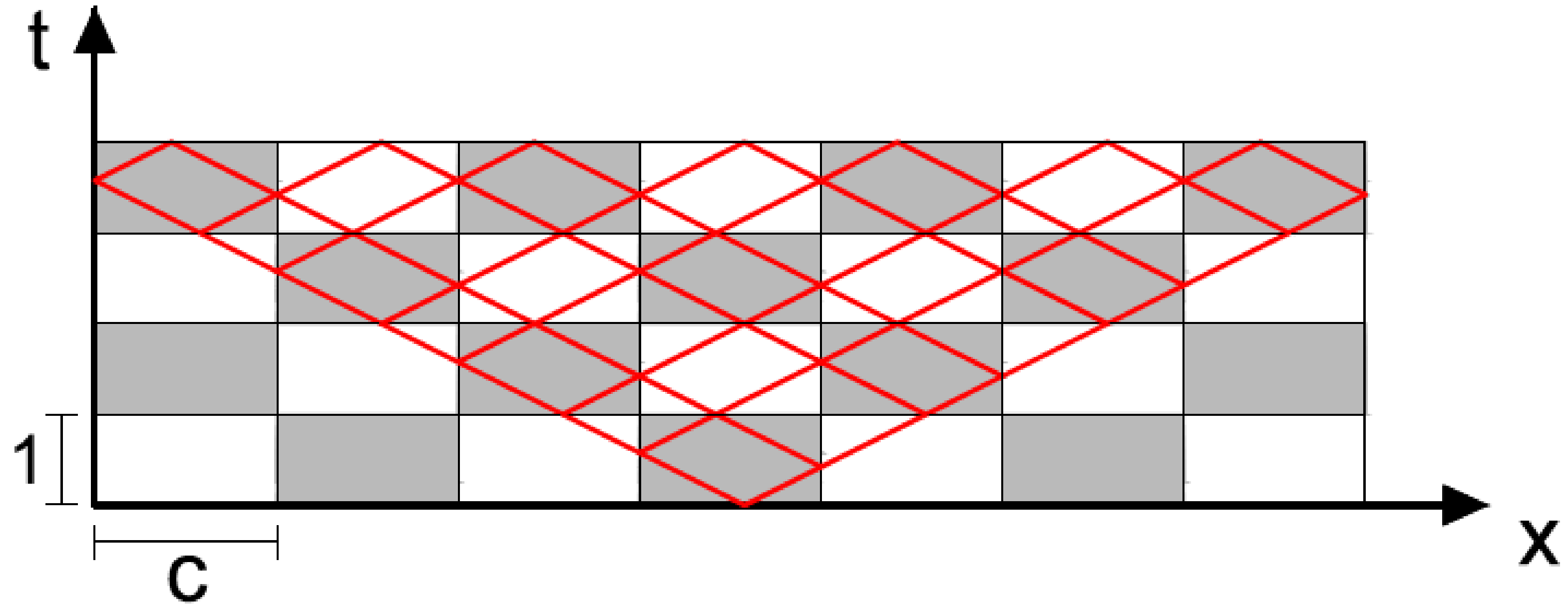
Curiously they found accumulations of the characteristic lines:



A bit like a shock but in a linear medium!

Field patterns in a space-time checkerboard

$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$



Field patterns are a new type of wave propagating along orderly patterns of characteristic lines!!!

Field patterns in a space-time checkerboard

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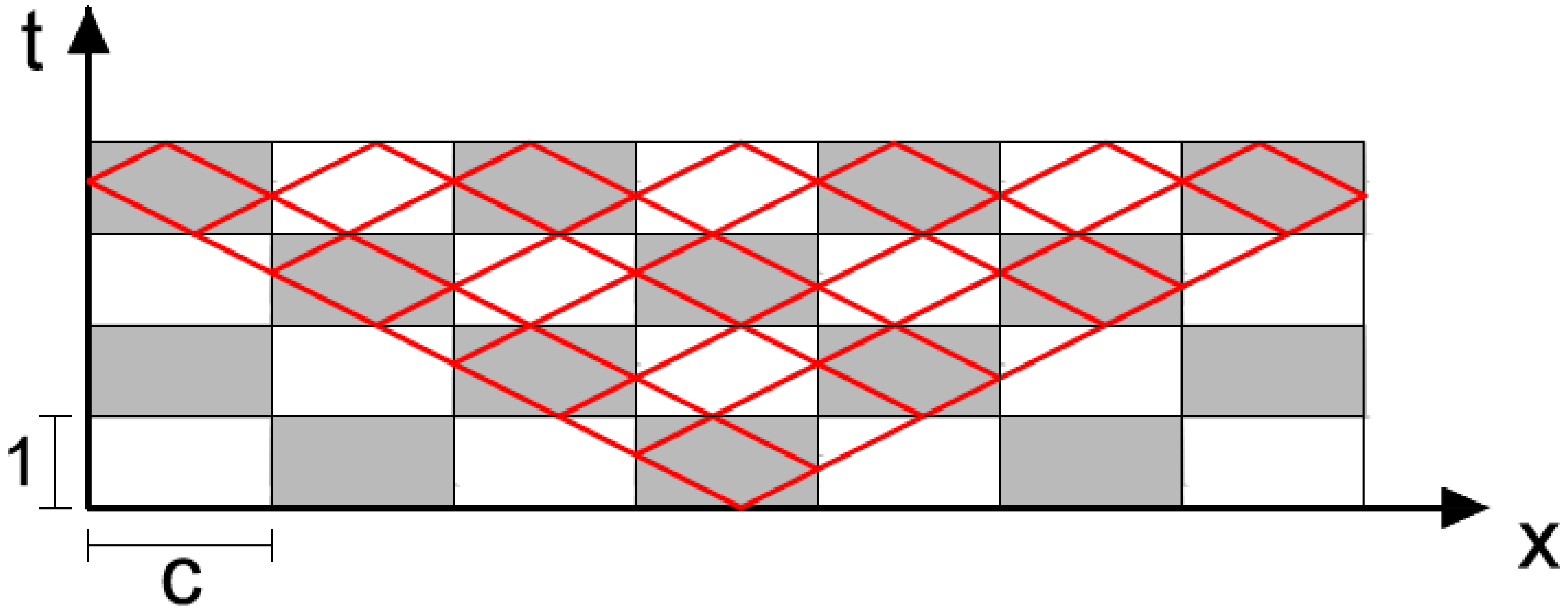
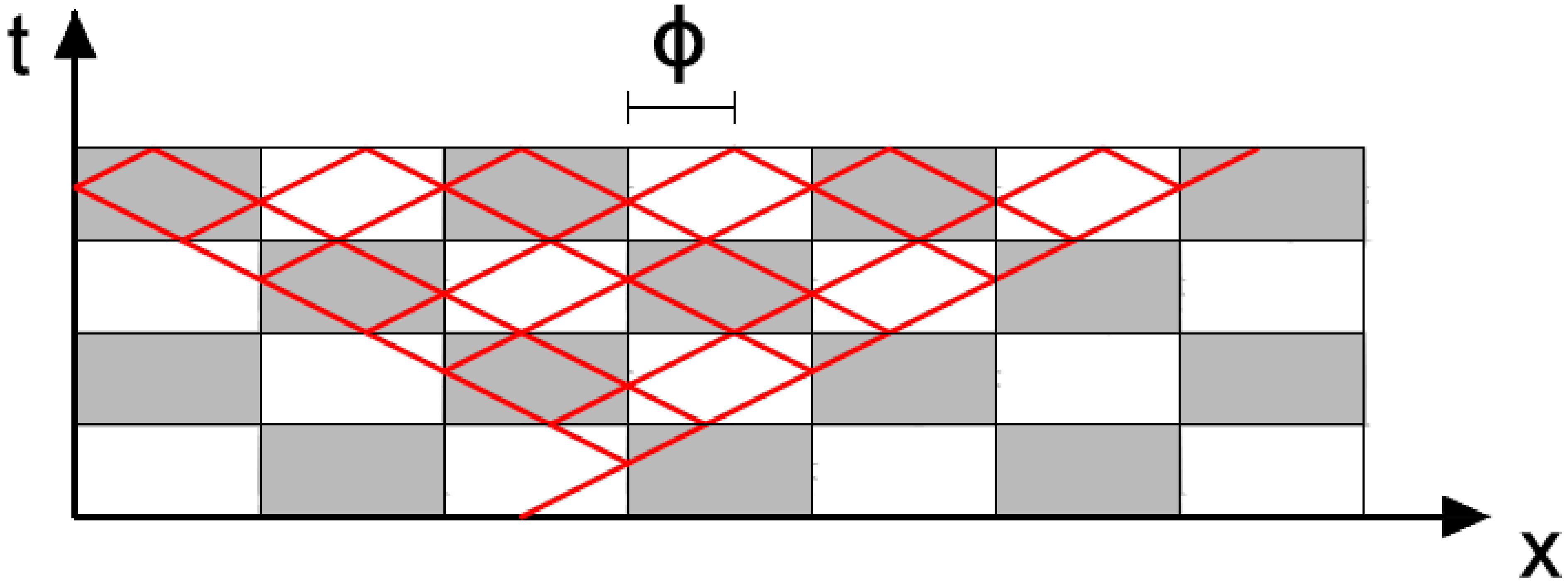


Families of field patterns

$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$



Families of field patterns

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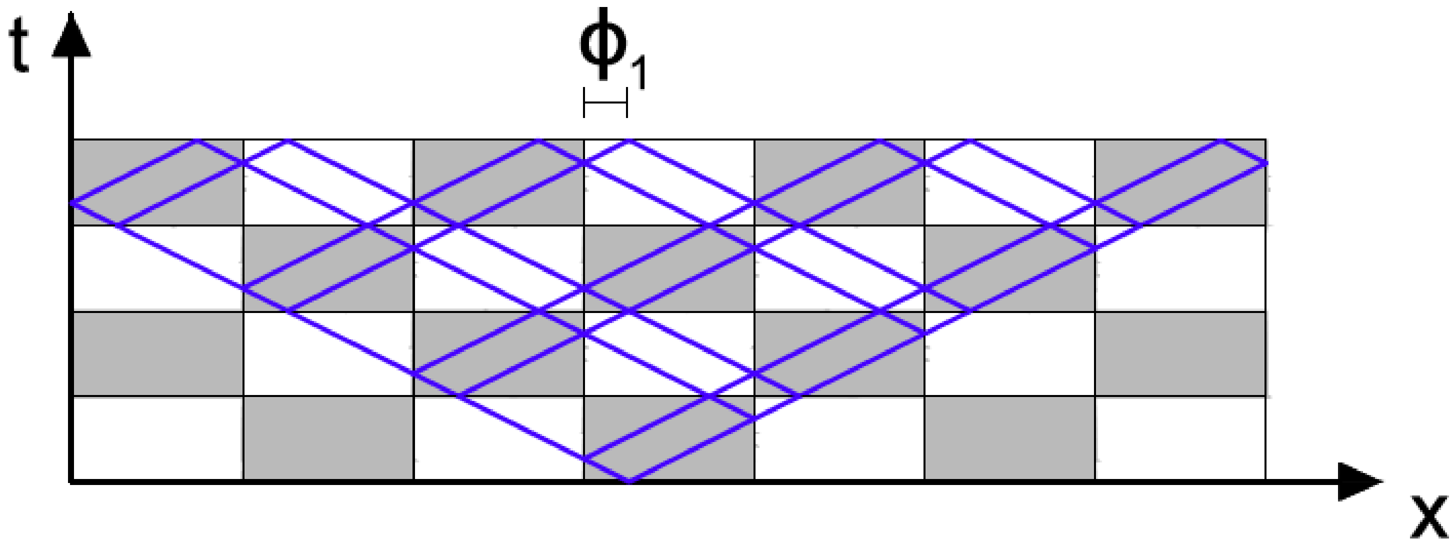
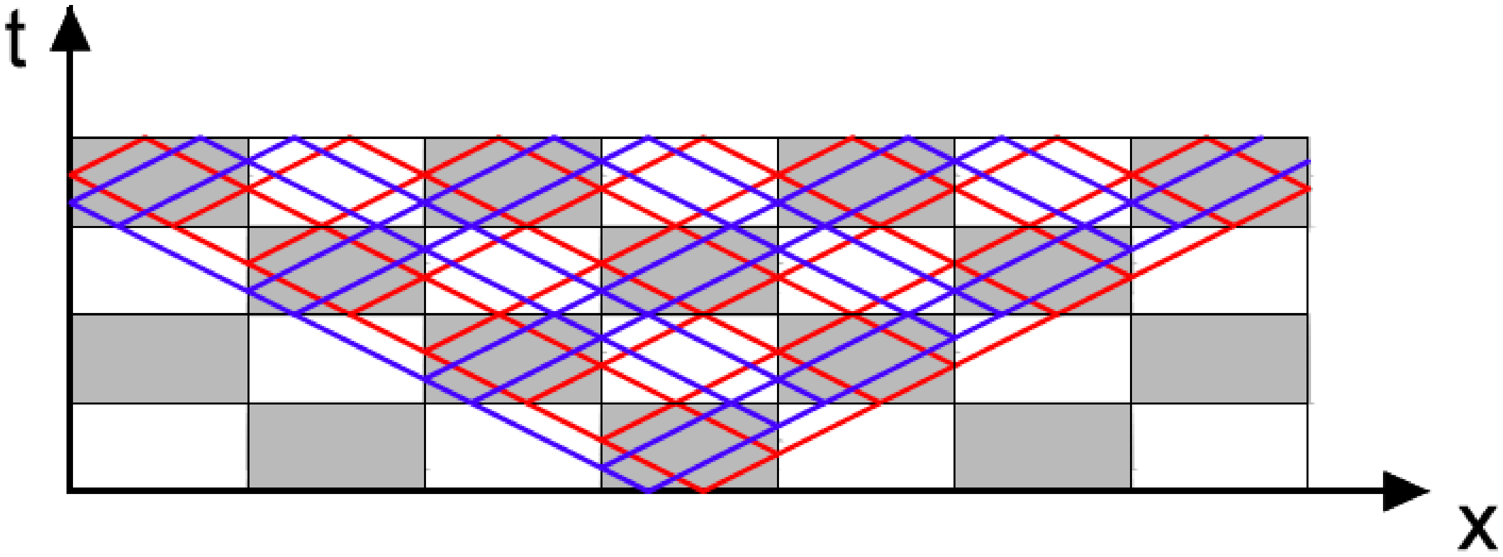


Field patterns are a new type of wave propagating along orderly patterns of characteristic lines which arise in specific space-time microstructures whose geometry in one spatial dimension plus time is somehow commensurate to the slope of the characteristic lines.

Multidimensional nature of field patterns

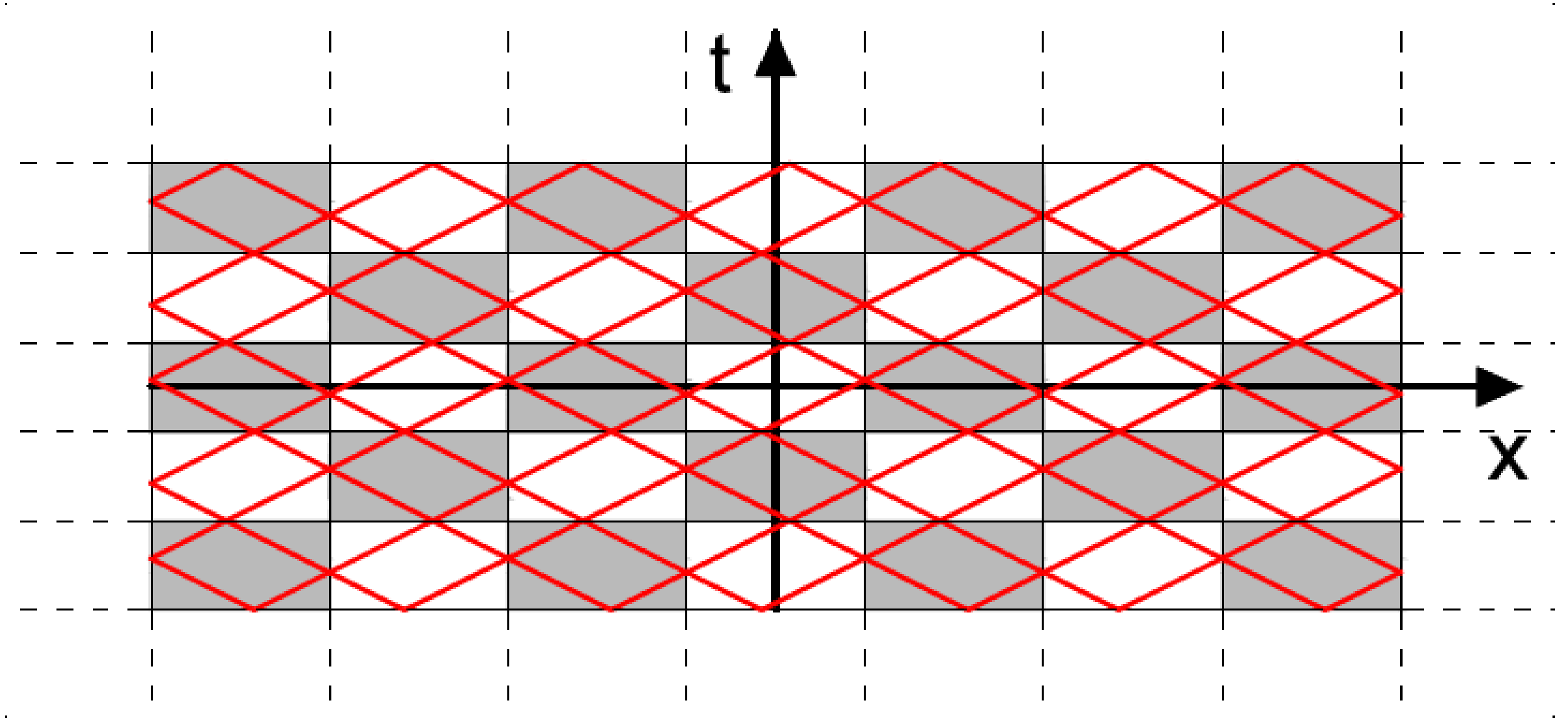


$$V(x,t) = \sum_{i=1}^{m} V_{\phi_i}(x,t)$$

Multidimensional space: $V(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{m} V_{\phi_i}(x_i, t)$

Multicomponent potential: $\mathbf{V}(x,t)$

PT–symmetry of field patterns

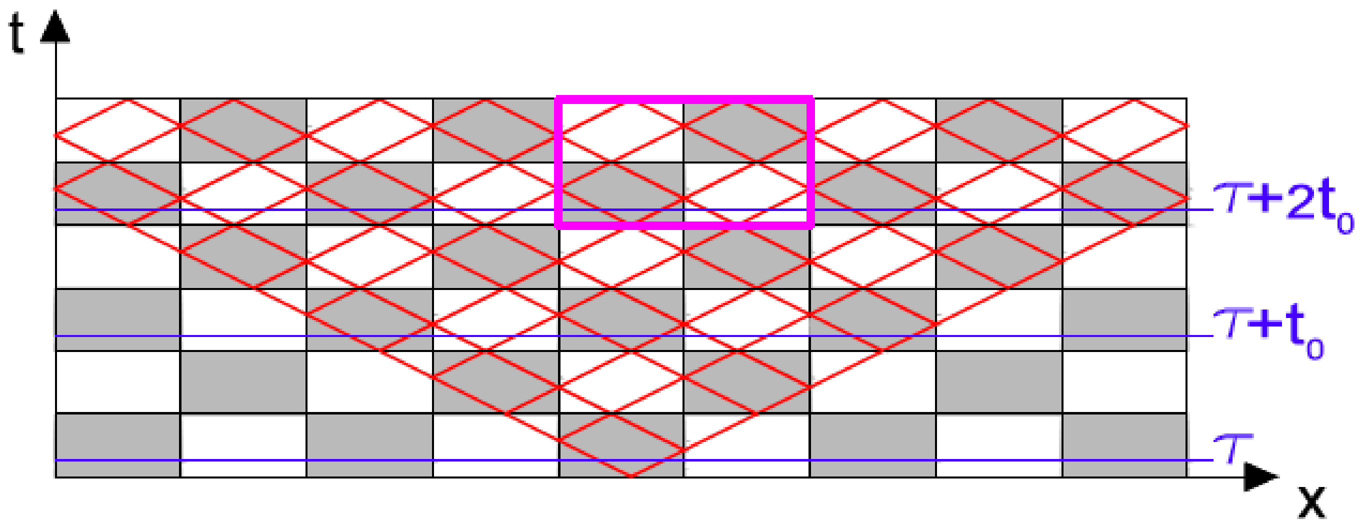


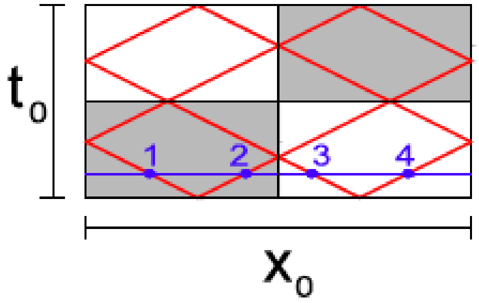
[Quantum physics, e.g., Bender and Boettcher, 1998,
Optics, e.g., Zyablovsky et al., 2014]

Unbroken PT–symmetry $\rightarrow$ real eigenvalues

Broken PT–symmetry $\rightarrow$ complex conjugate eigenvalues

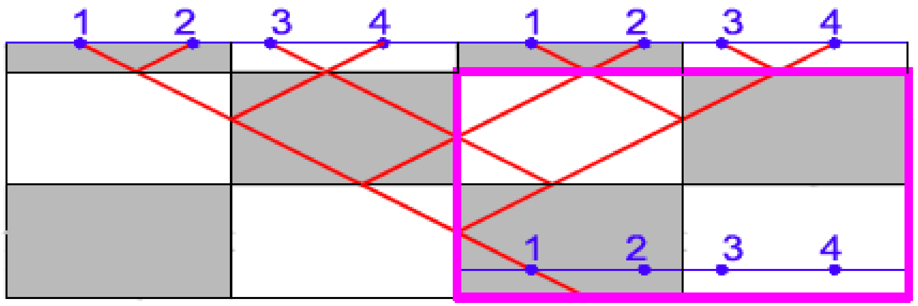
The transfer matrix





$$j(k,m,n+1) = \sum_{k',m'} T_{(k,m),(k,'m')} j(k',m',n)$$

$$T_{(k,m),(k,'m')} = G_{k,k'}(m-m')$$

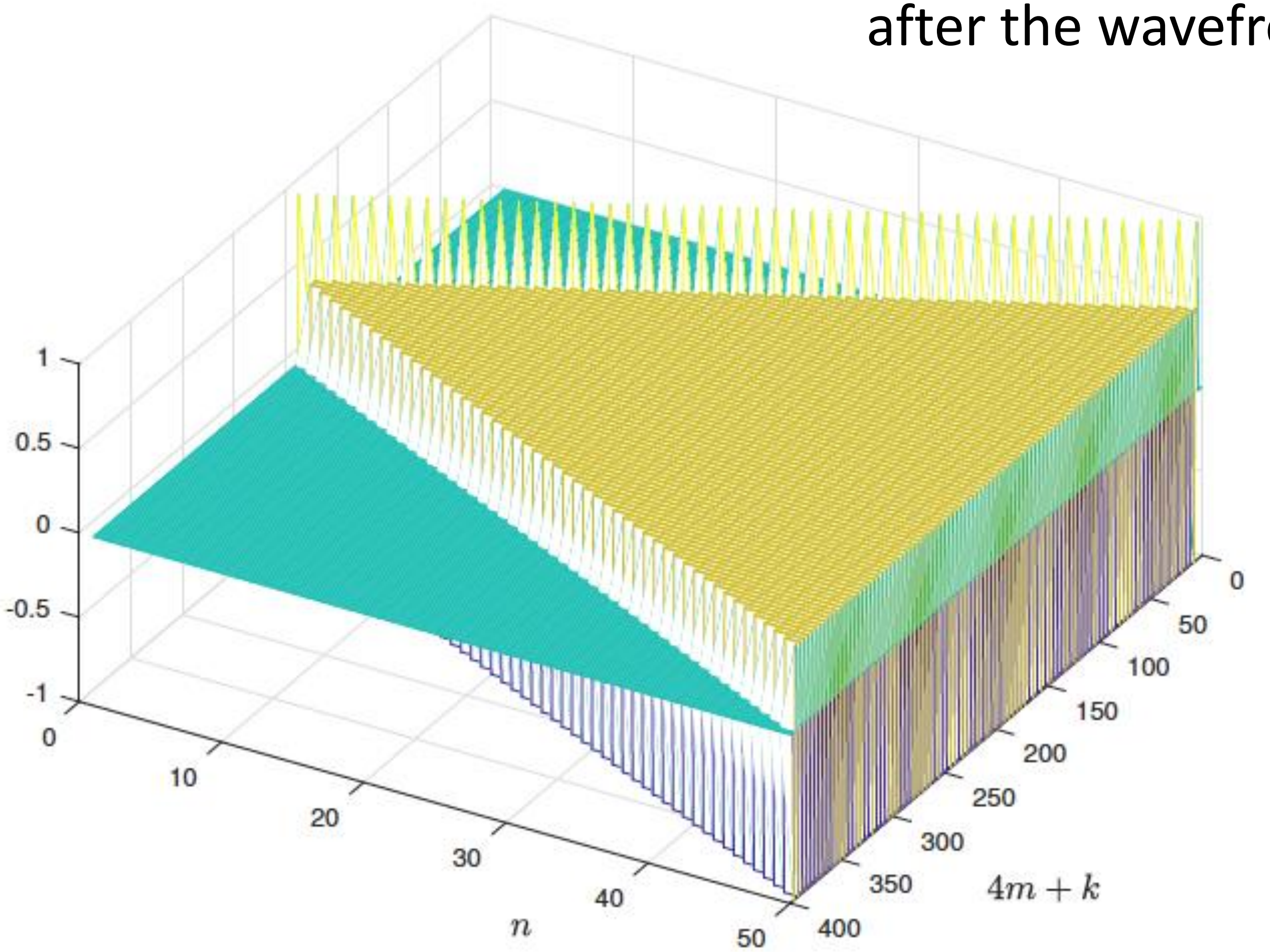


T depends only on γ_i

T is PT–symmetric

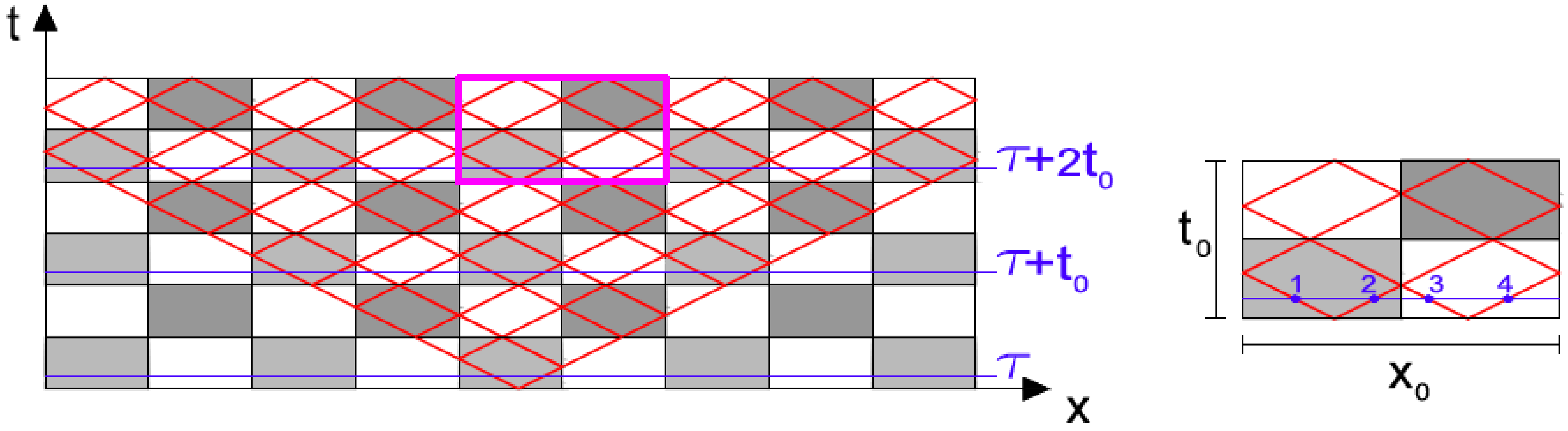
Evolution in time of the current distribution

Note: oscillations continue after the wavefront!



Three–component space–time checkerboard

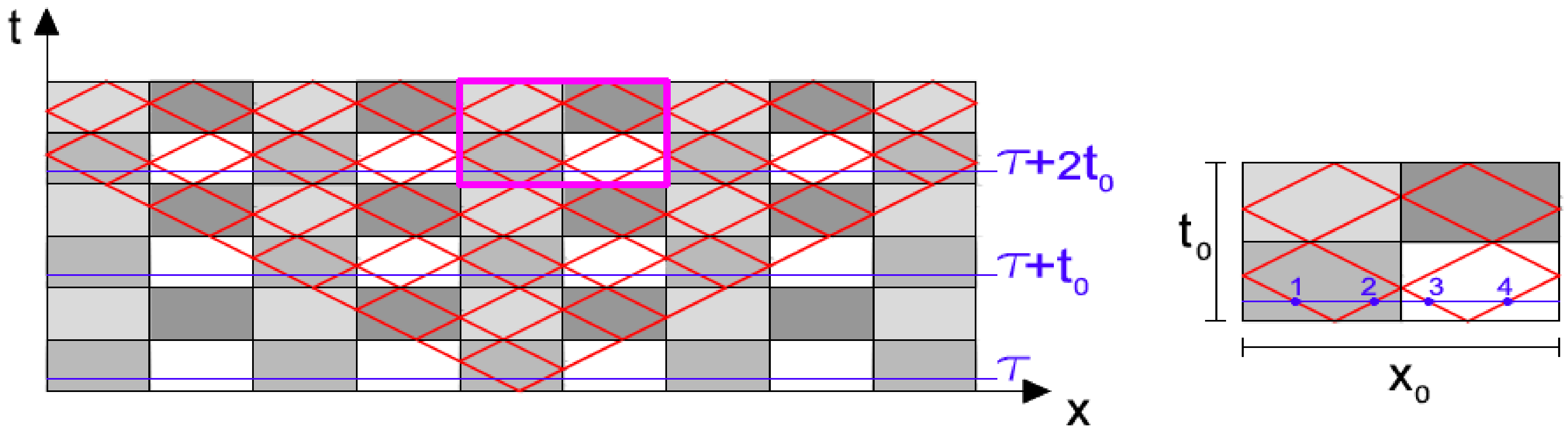
$$c_1 = c_2 = c_3$$



UNBROKEN PT–symmetry $\Rightarrow$ Only propagating modes

Four–component space–time checkerboard

$$c_1 = c_2 = c_3 = c_4$$

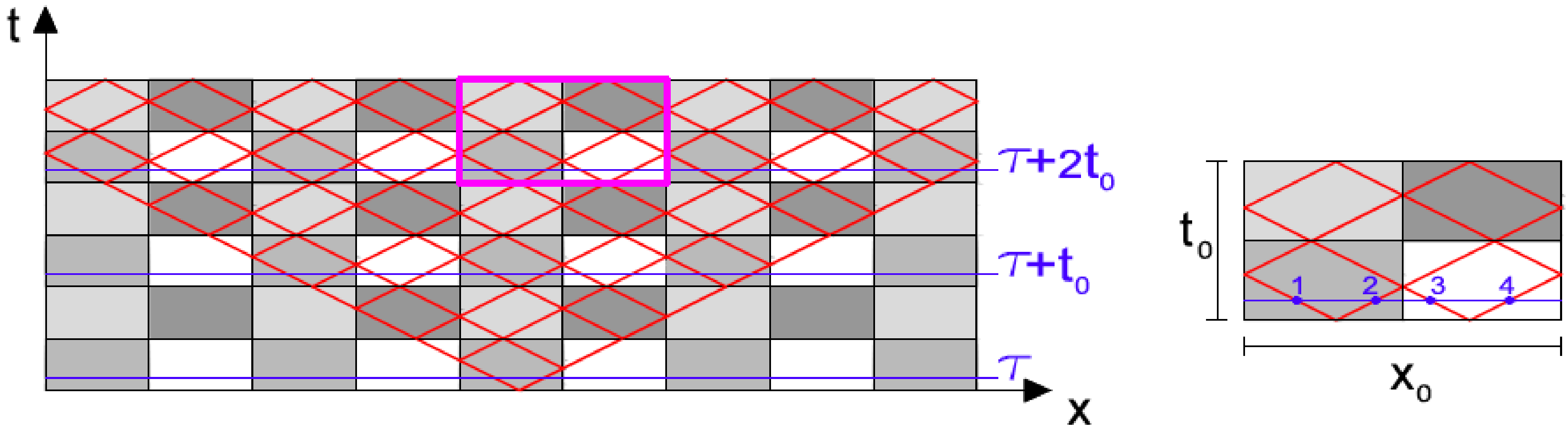


For some combinations of γ_1, γ_2, γ_3, γ_4 : UNBROKEN PT–symmetry

For other combinations of γ_1, γ_2, γ_3, γ_4 : BROKEN PT–symmetry

Four–component space–time checkerboard

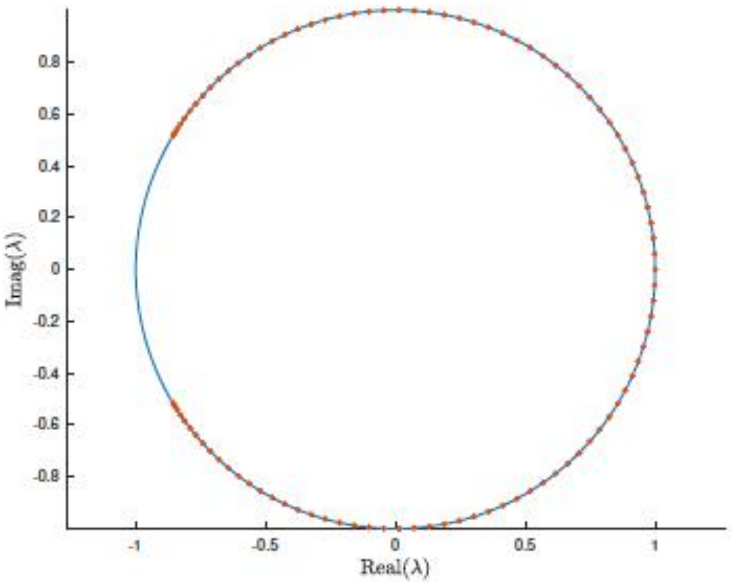
$$c_1 = c_2 = c_3 = c_4$$

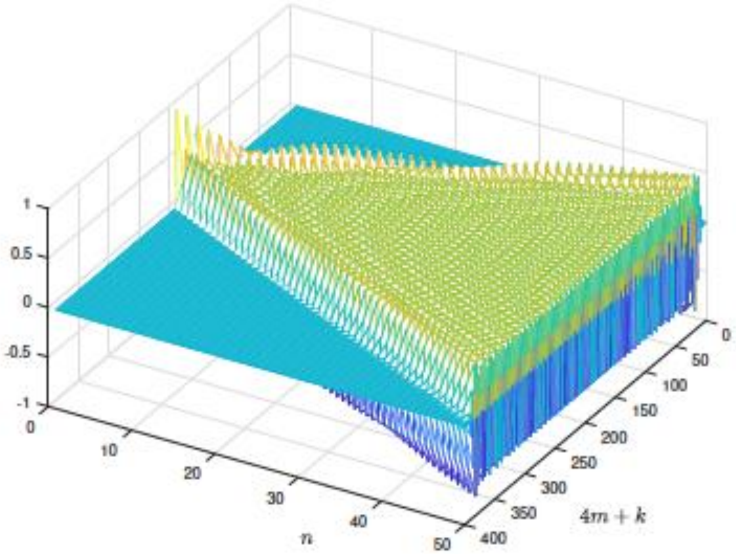


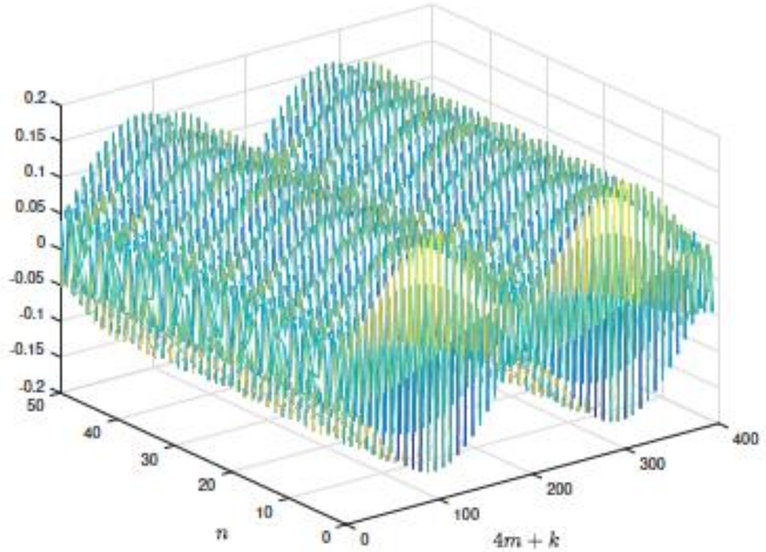
For some combinations of γ_1, γ_2, γ_3, γ_4 : UNBROKEN PT–symmetry

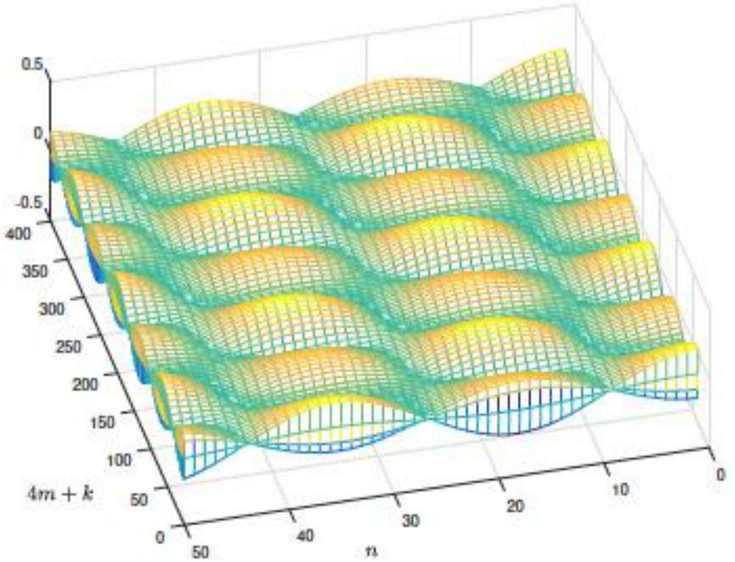
For other combinations of γ_1, γ_2, γ_3, γ_4 : BROKEN PT–symmetry

Unbroken PT–symmetry for the four–phase checkerboard

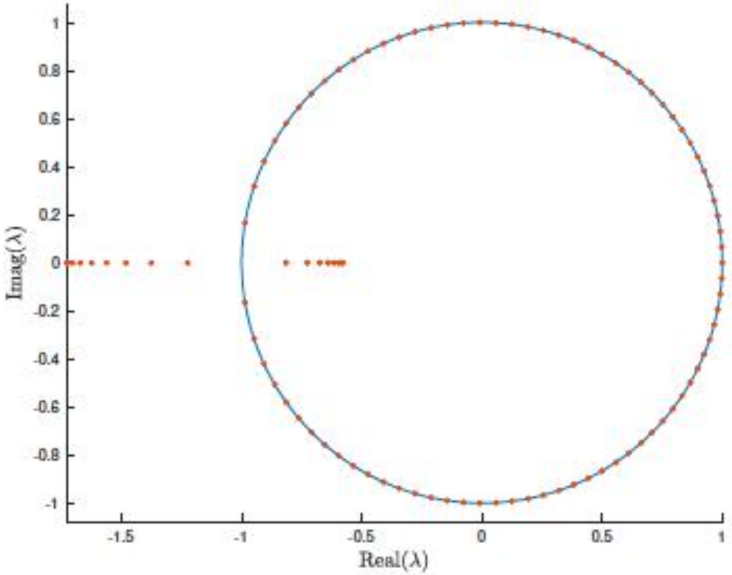


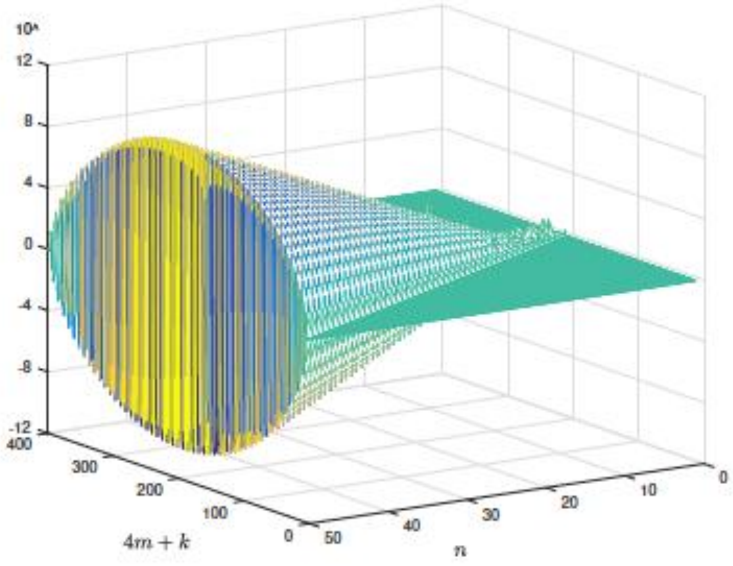






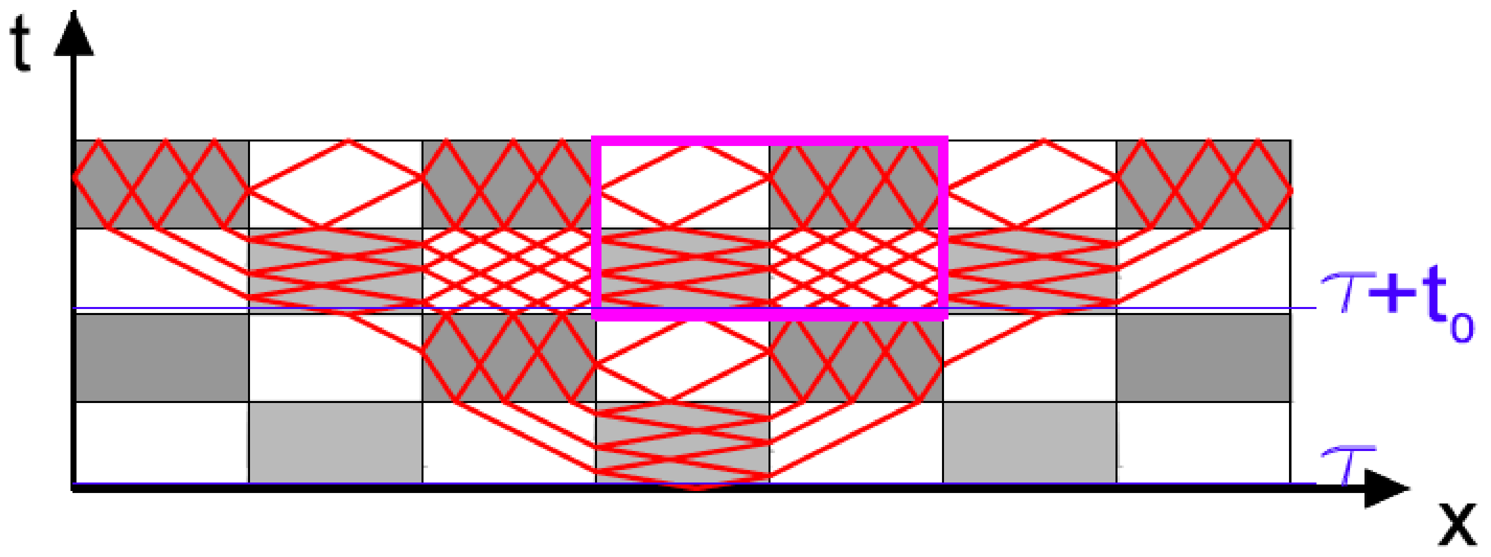
Broken PT–symmetry for the four–phase checkerboard





Three–phase space–time checkerboard

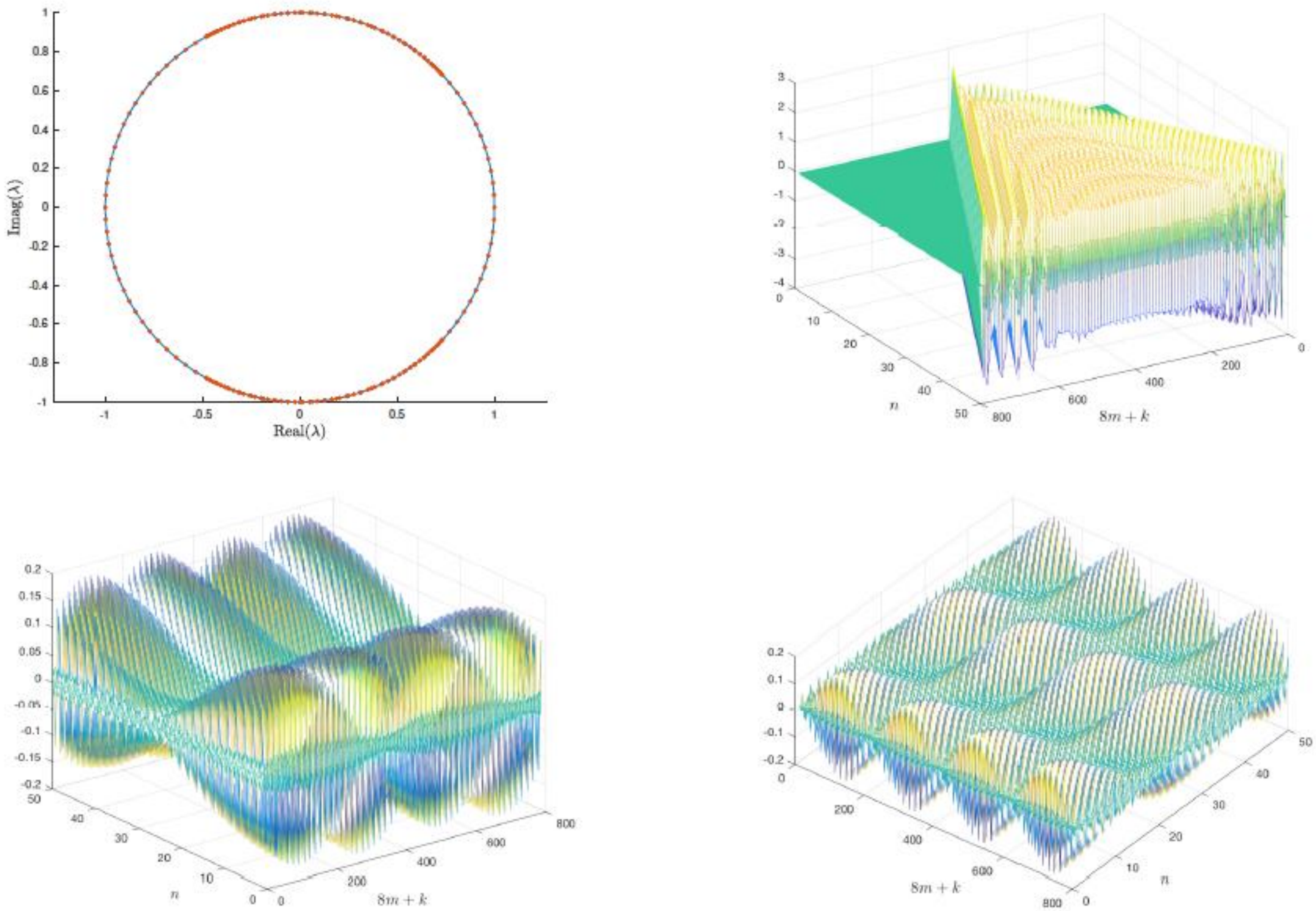
$$c_2/c_1 = c_1/c_3 = 3$$



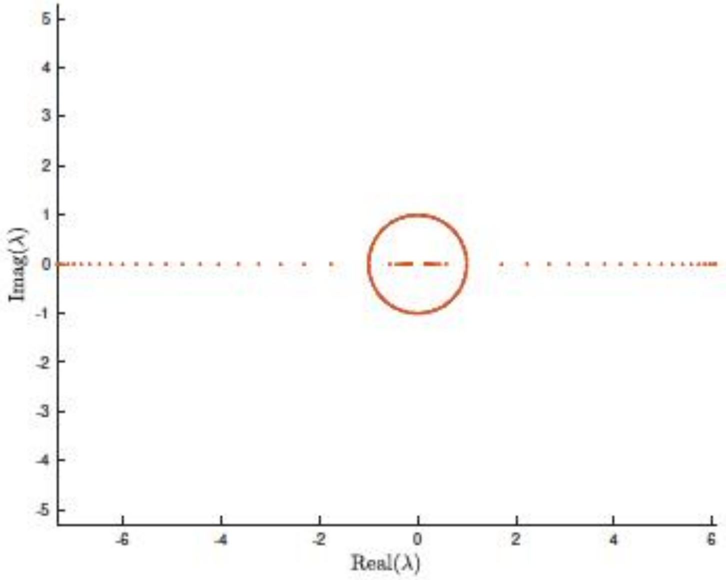
For some combinations of γ_1, γ_2, γ_3 : UNBROKEN PT–symmetry

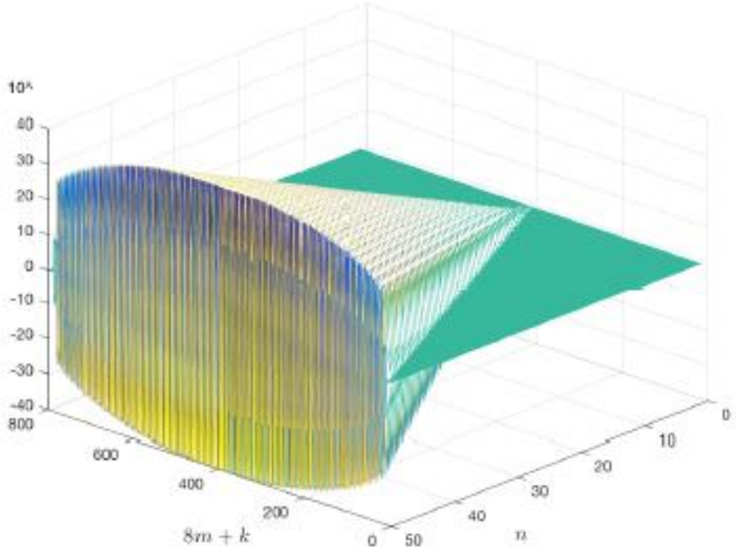
For other combinations of γ_1, γ_2, γ_3 : BROKEN PT–symmetry

Unbroken PT–symmetry for the three–phase checkerboard



Broken PT–symmetry for the three–phase checkerboard

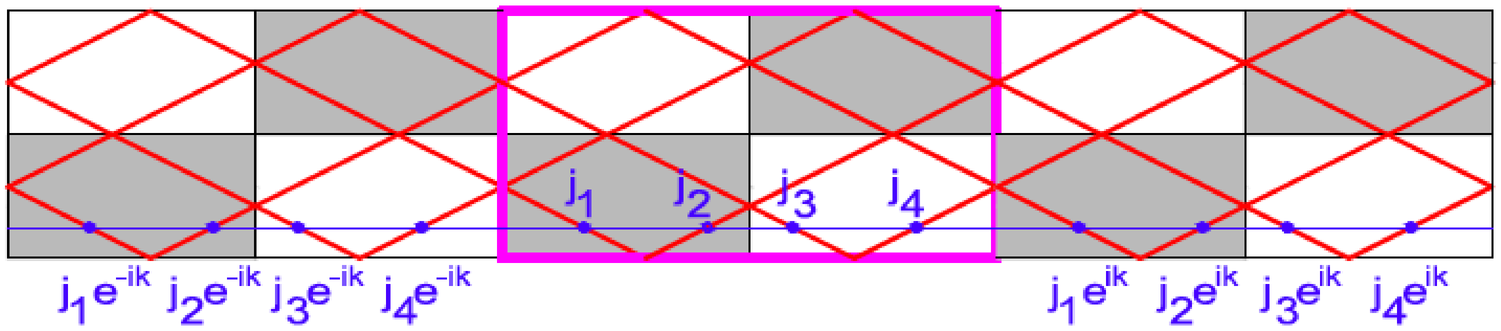




Bloch–Floquet theory applied to field patterns

Periodicity with respect to x:

$$j(l, m+s, n) = \exp(\mathrm{i}ks)\, j(l, m, n)$$



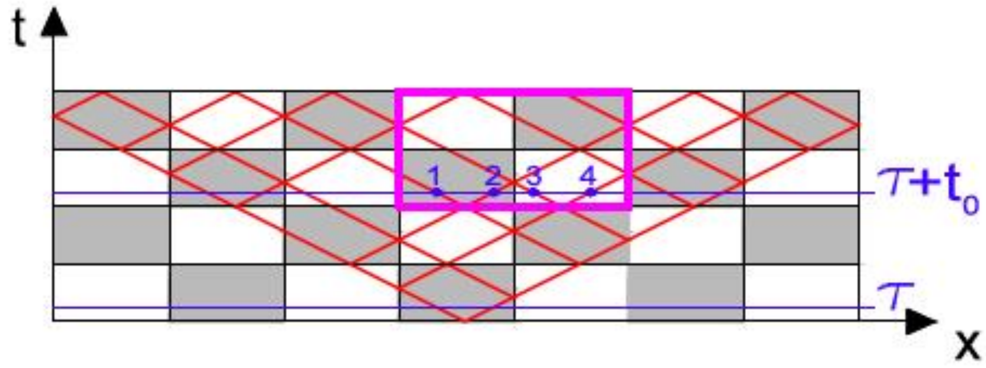
Periodicity with respect to t:

$$j(l, m, n+q) = \exp(\mathrm{i}\,\omega\, q) j(l, m, n)$$

Recall: $j(l, m, n+q) = \lambda^q\, j(l, m, n)$, then

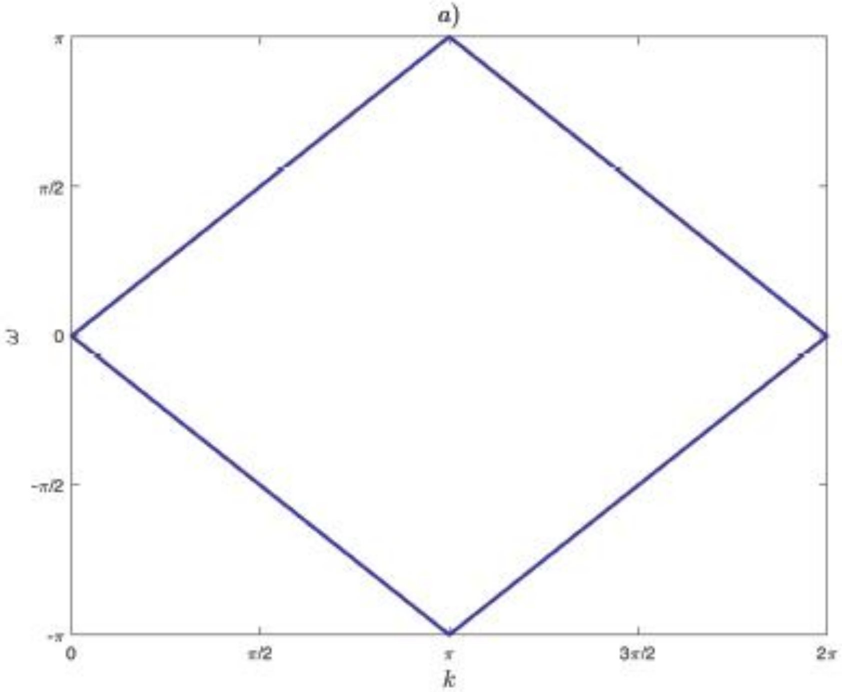
Dispersion relation : $\boxed{\lambda(k) = \exp(\mathrm{i}\,\omega)}$

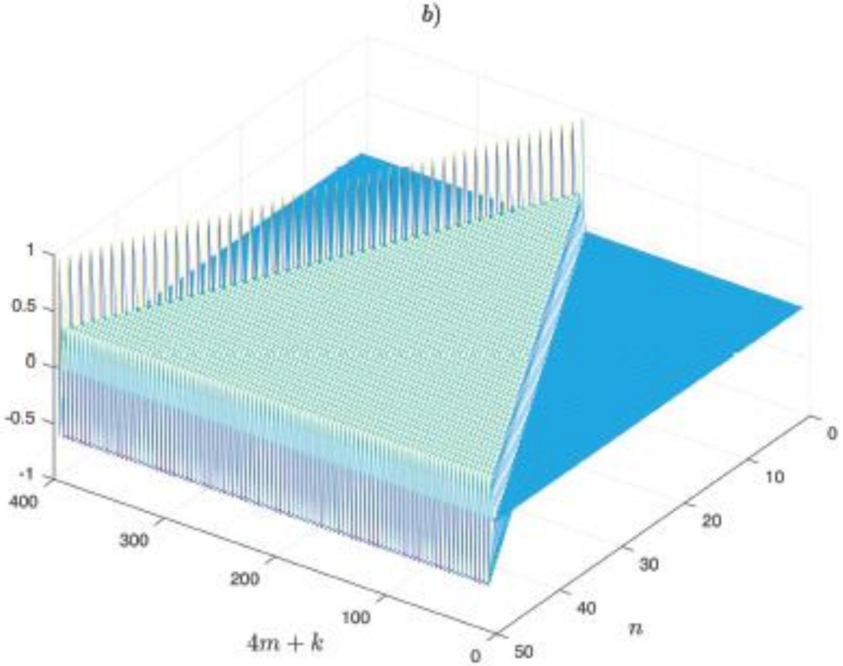
Dispersion diagram for the two–phase checkerboard



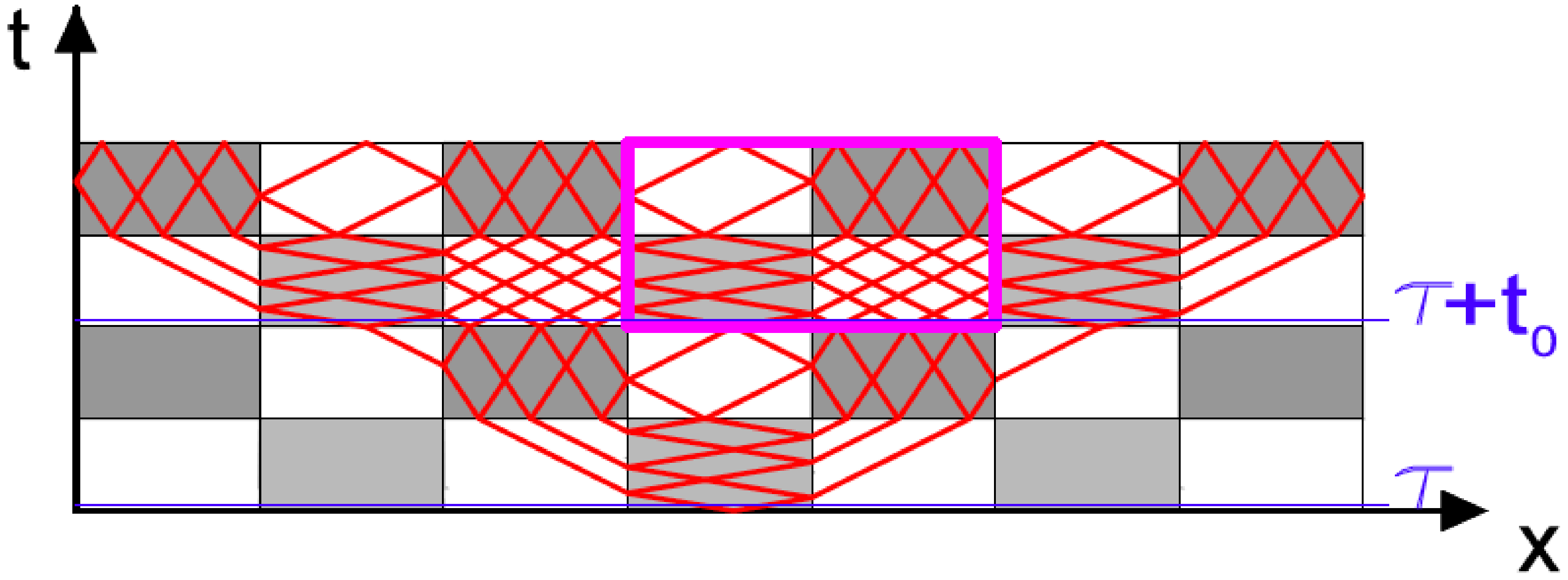
$$\lambda(k) = \exp(\pm \mathrm{i}k)$$

Dispersion relation: $\omega = \pm k$

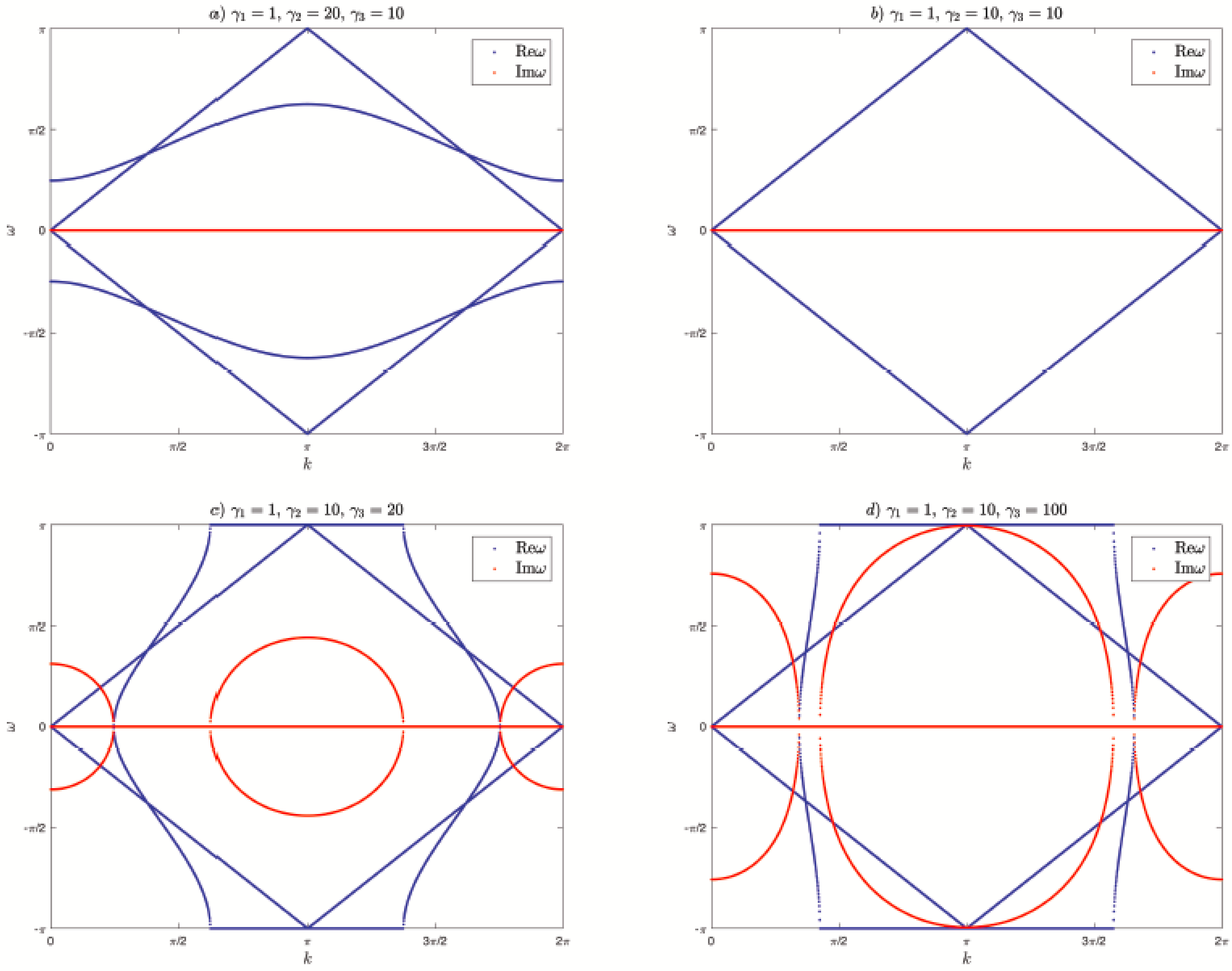




Three–phase checkerboard with phases having speed in a certain ratio



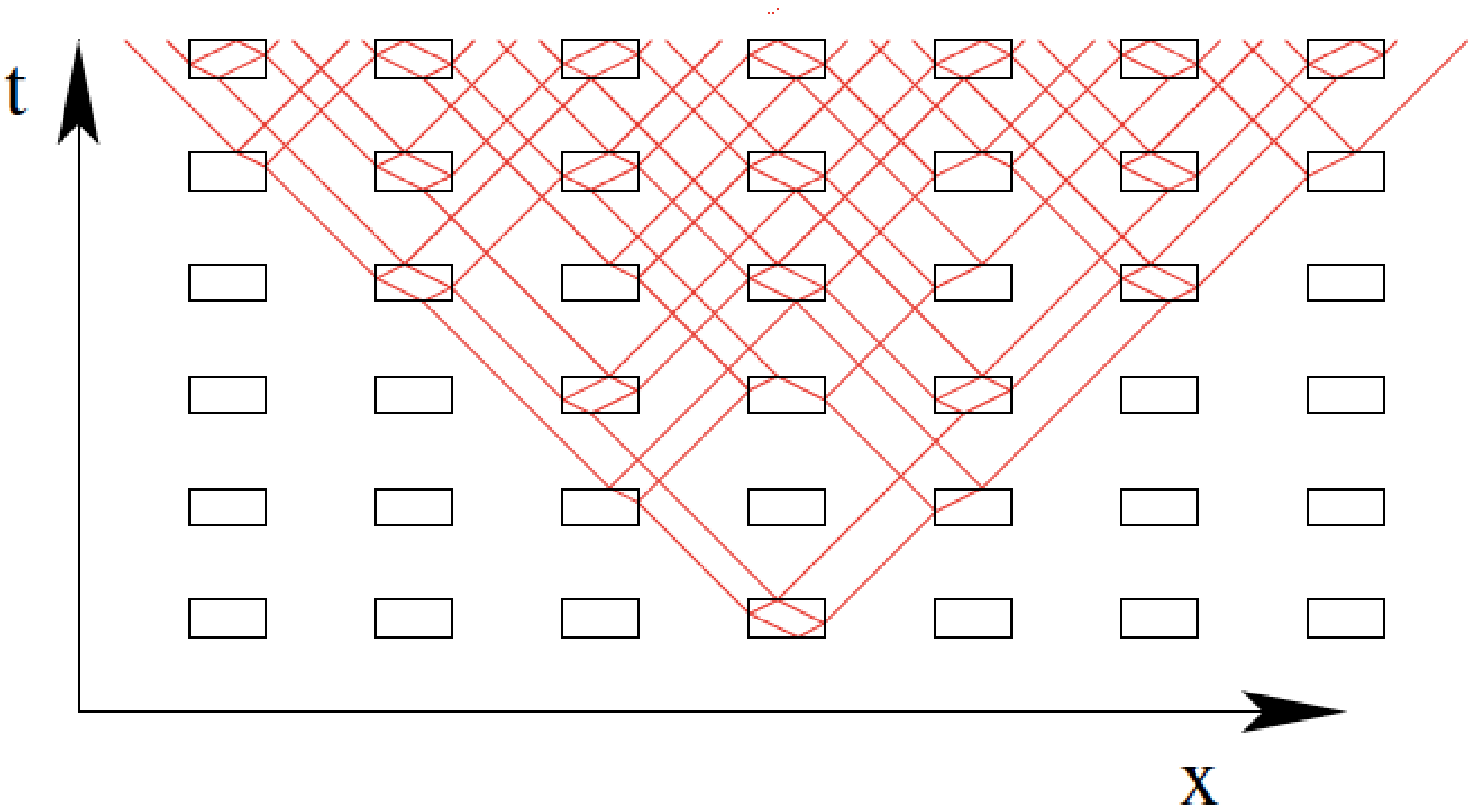
Dispersion diagrams for the three–phase checkerboard



Bloch Waves are:

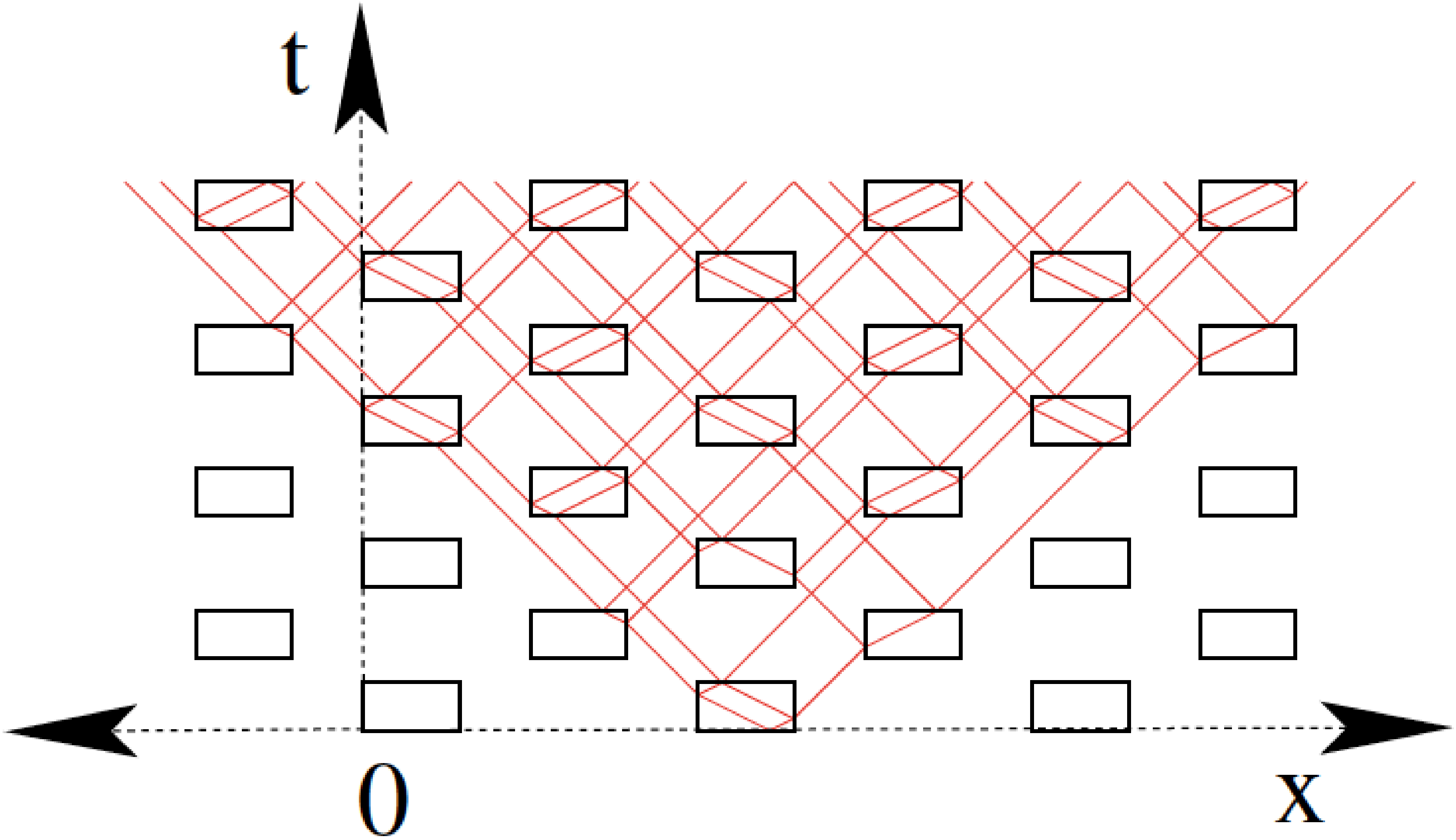
Infinitely Degenerate!

What about other field pattern geometries?



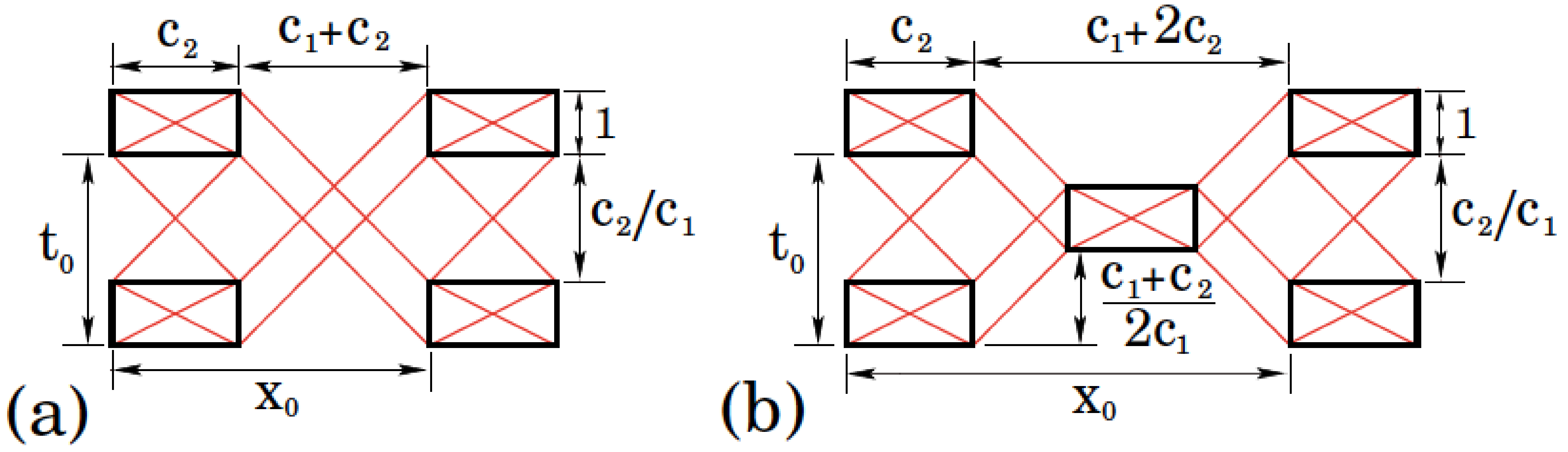
Characteristic lines lie on a pattern–the field pattern.
Note the P-T symmetry of the microstructure.

Alternatively one can have staggered inclusions:

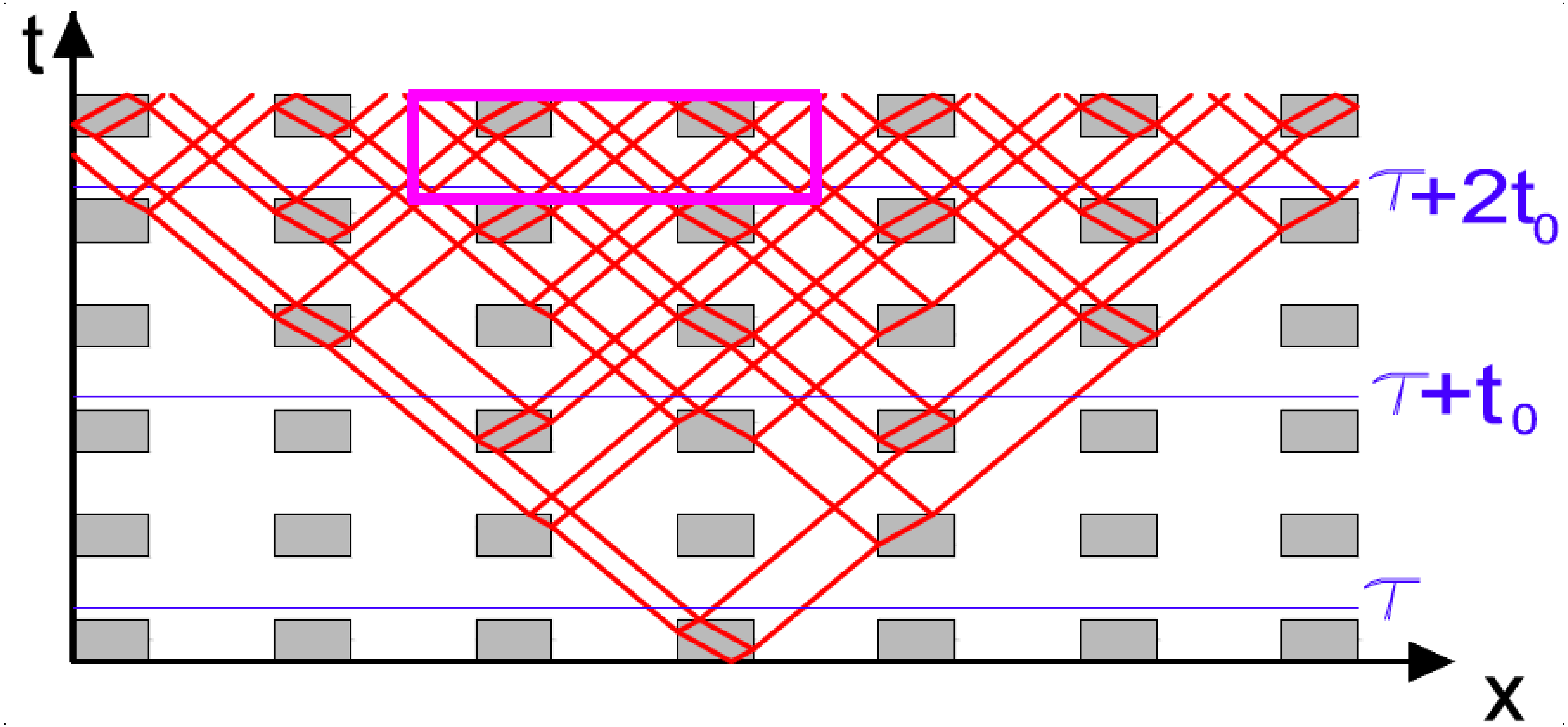


Again, note the P-T symmetry of the microstructure.

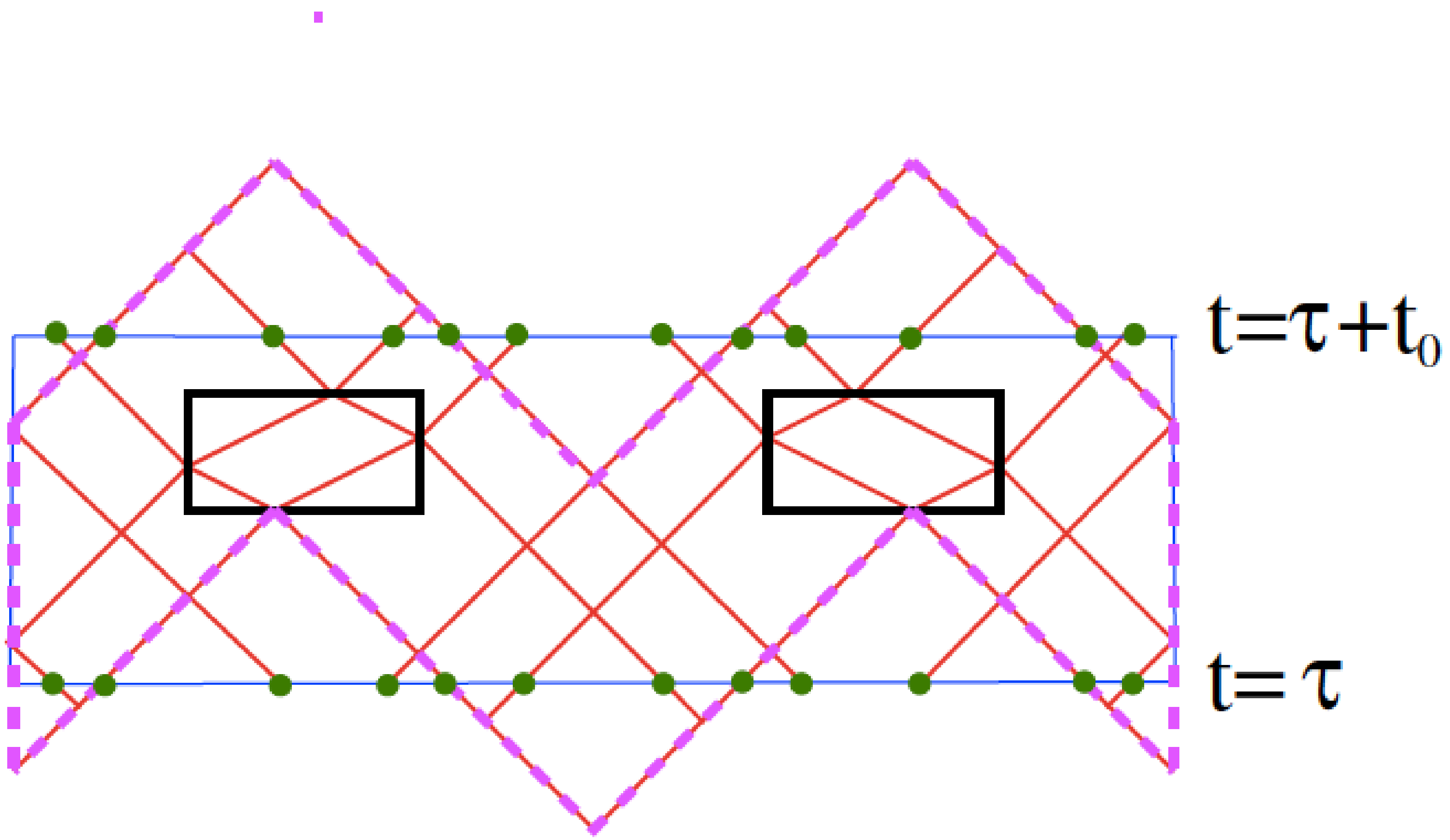
Geometry: Relation to Characteristic Lines



Space–time microstructure with rectangular inclusions



Numerical results: Transfer Matrix

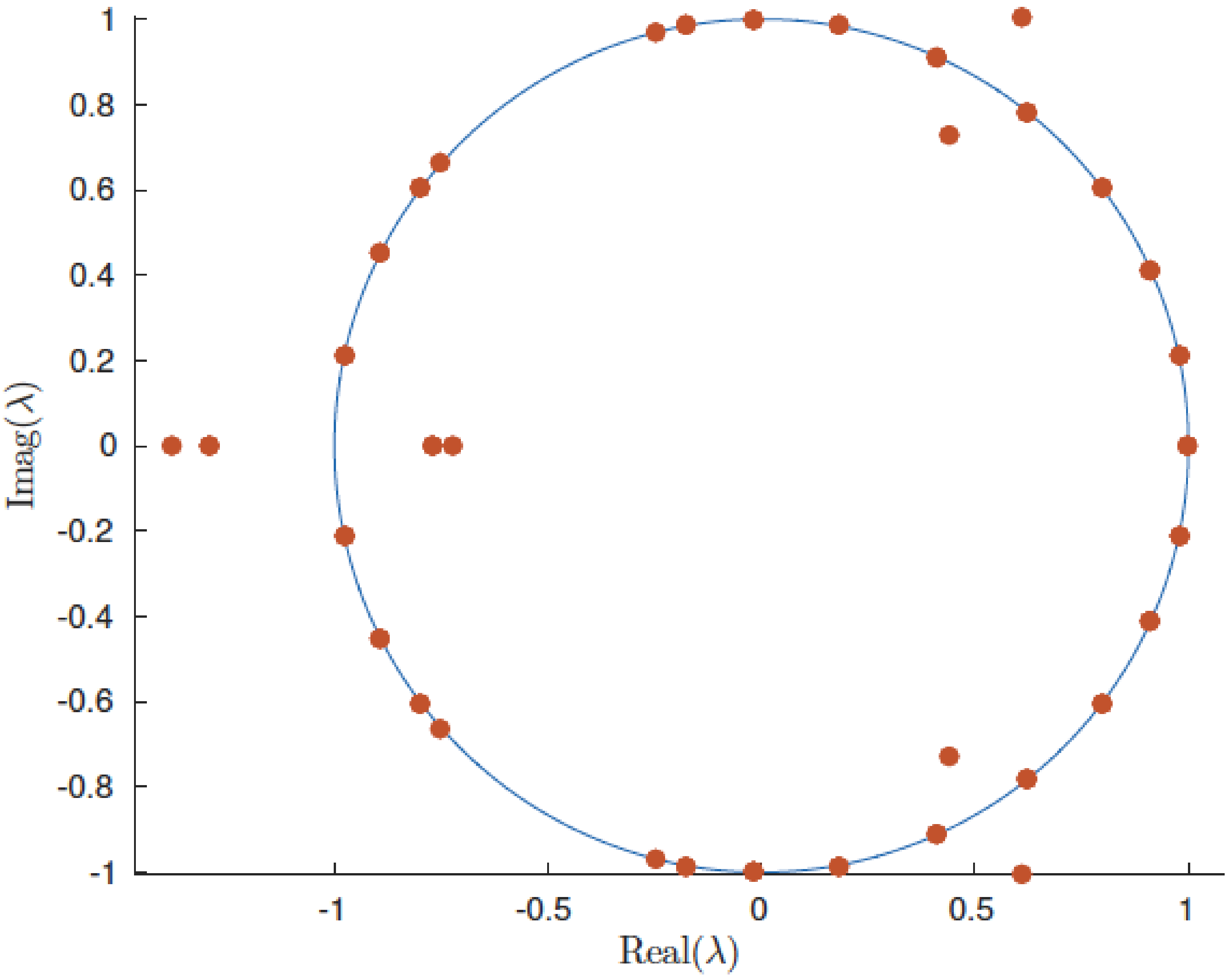


Note: The period cell of the field pattern is twice that of the microstructure!

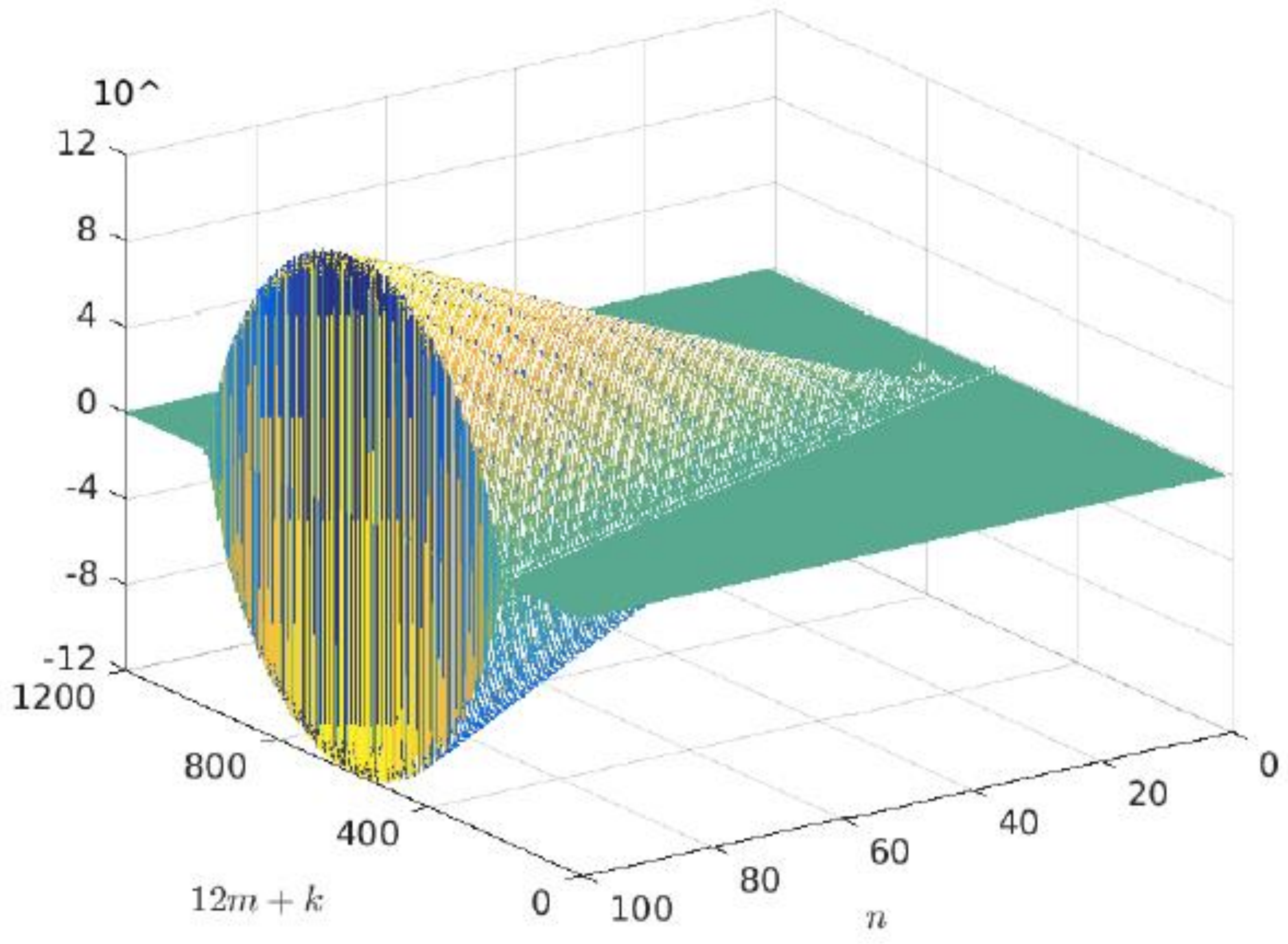
$$j(k, m, n+1) = \sum_{k',m'} T_{(k,m),(k,'m')} j(k', m', n)$$

$$T_{(k,m),(k,'m')} = G_{k,k'}(m - m')$$

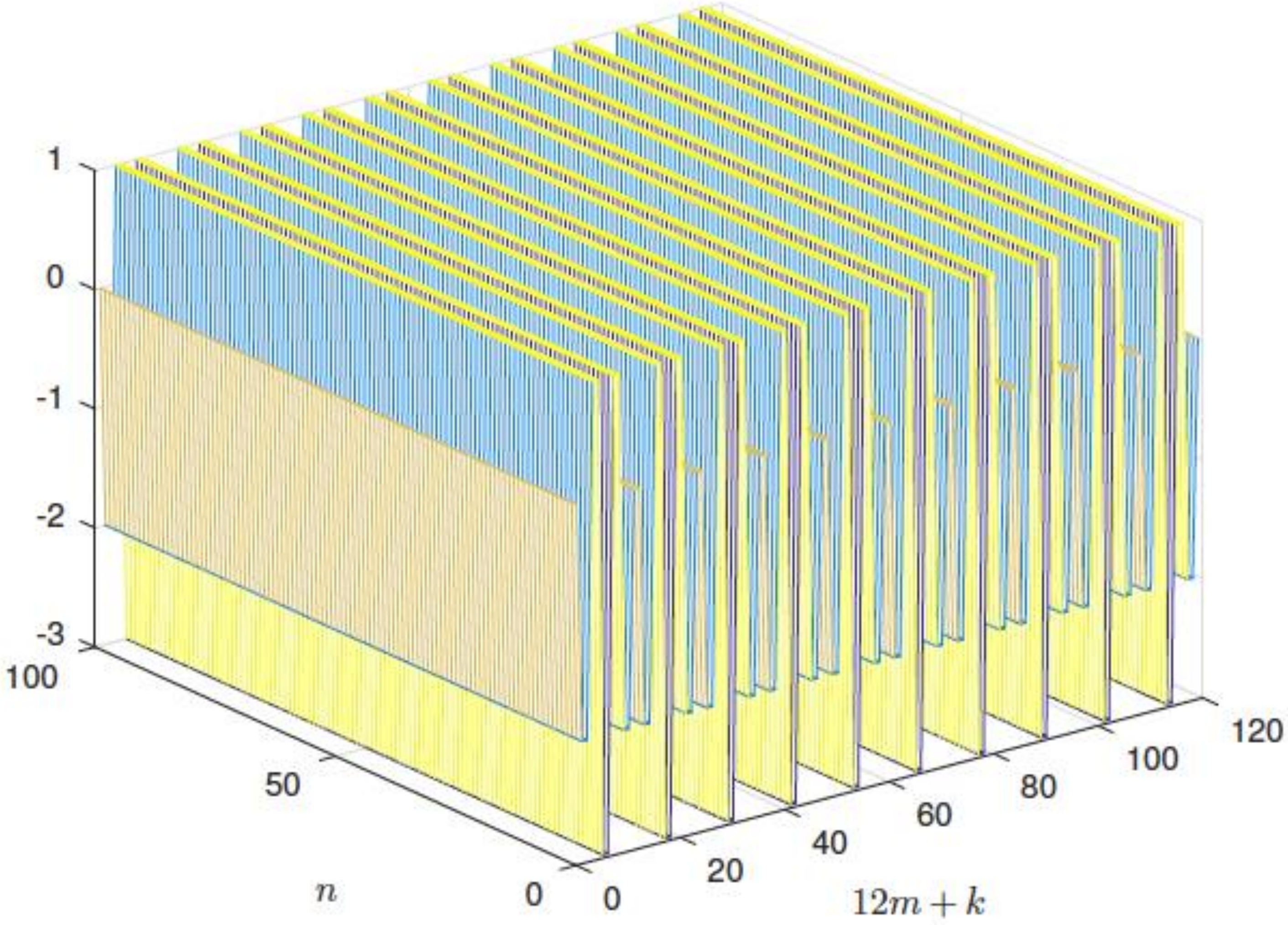
Eigenvalues of the transfer matrix



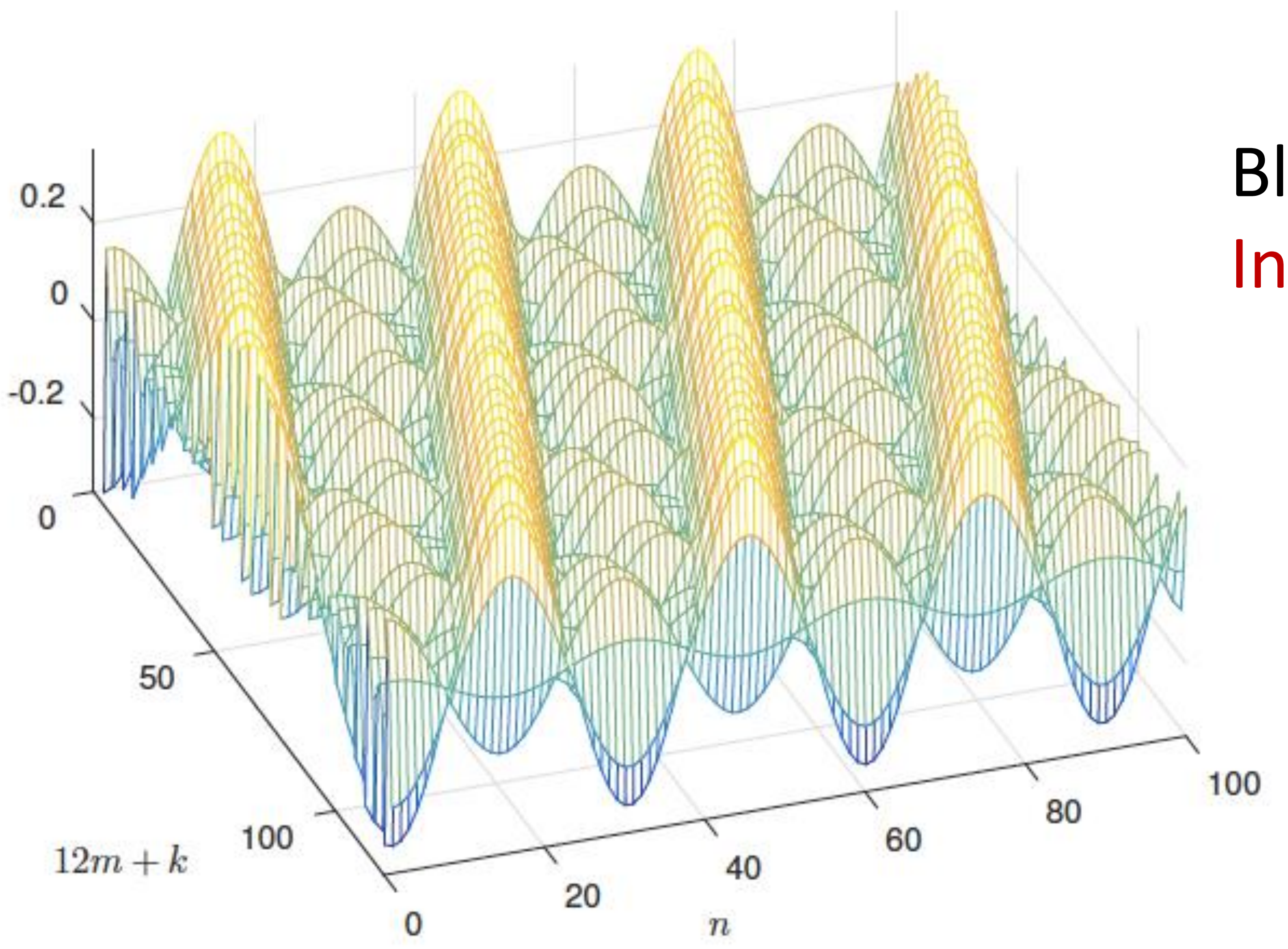
Blow up



Periodic solution



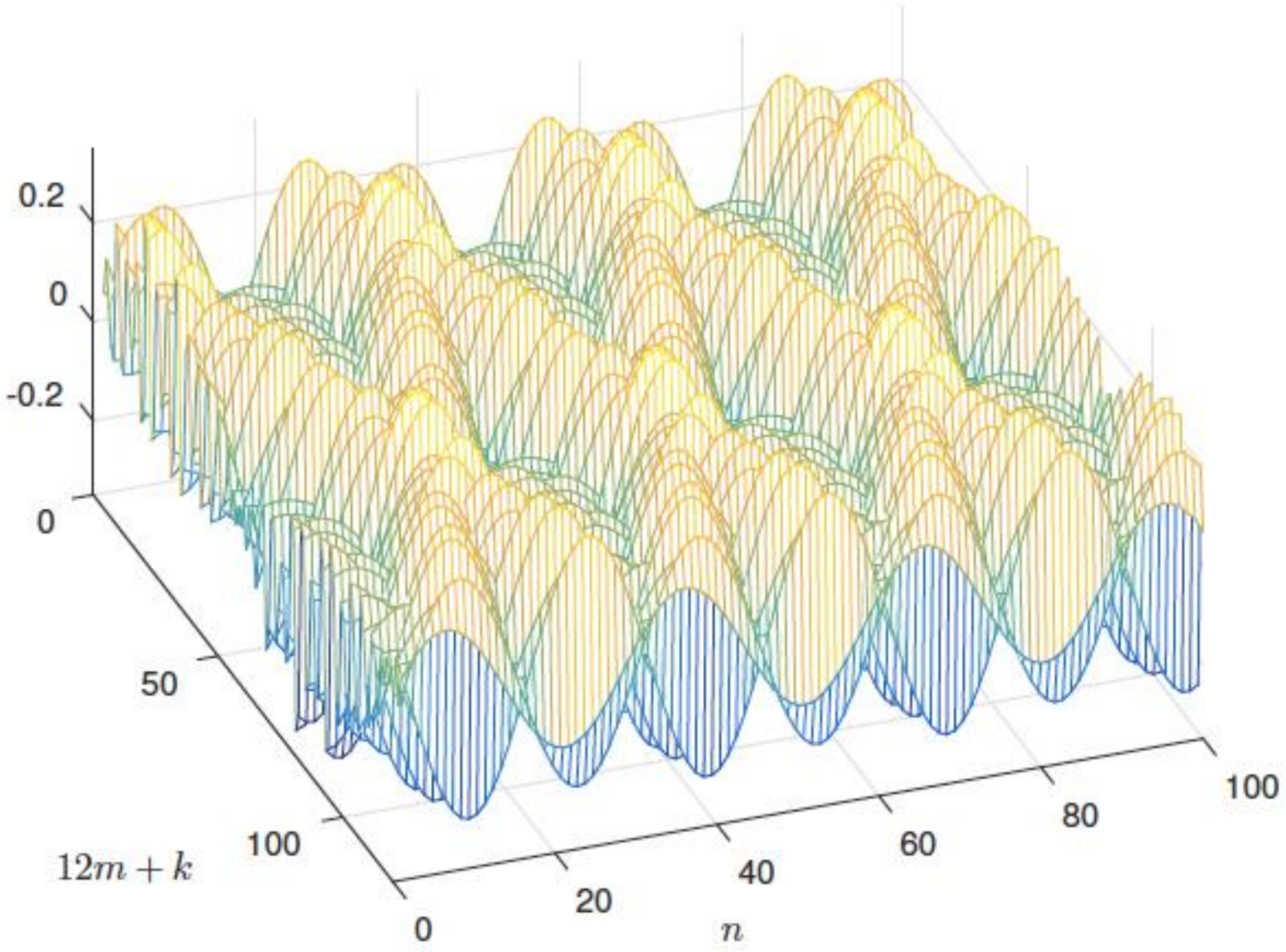
An example of a solution that does not blow up



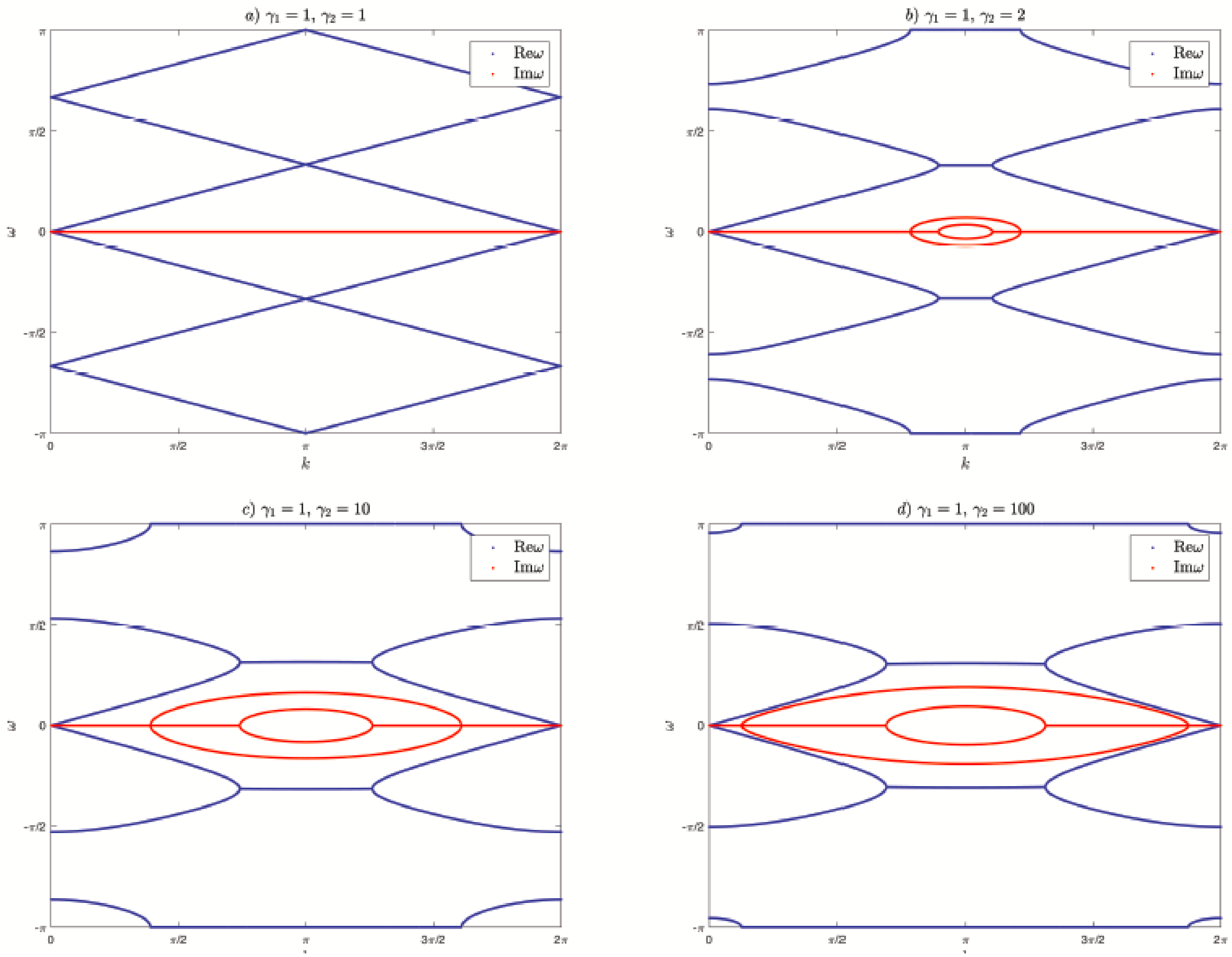
Bloch Wave:

Infinitely Degenerate!

One more solution that does not blow up

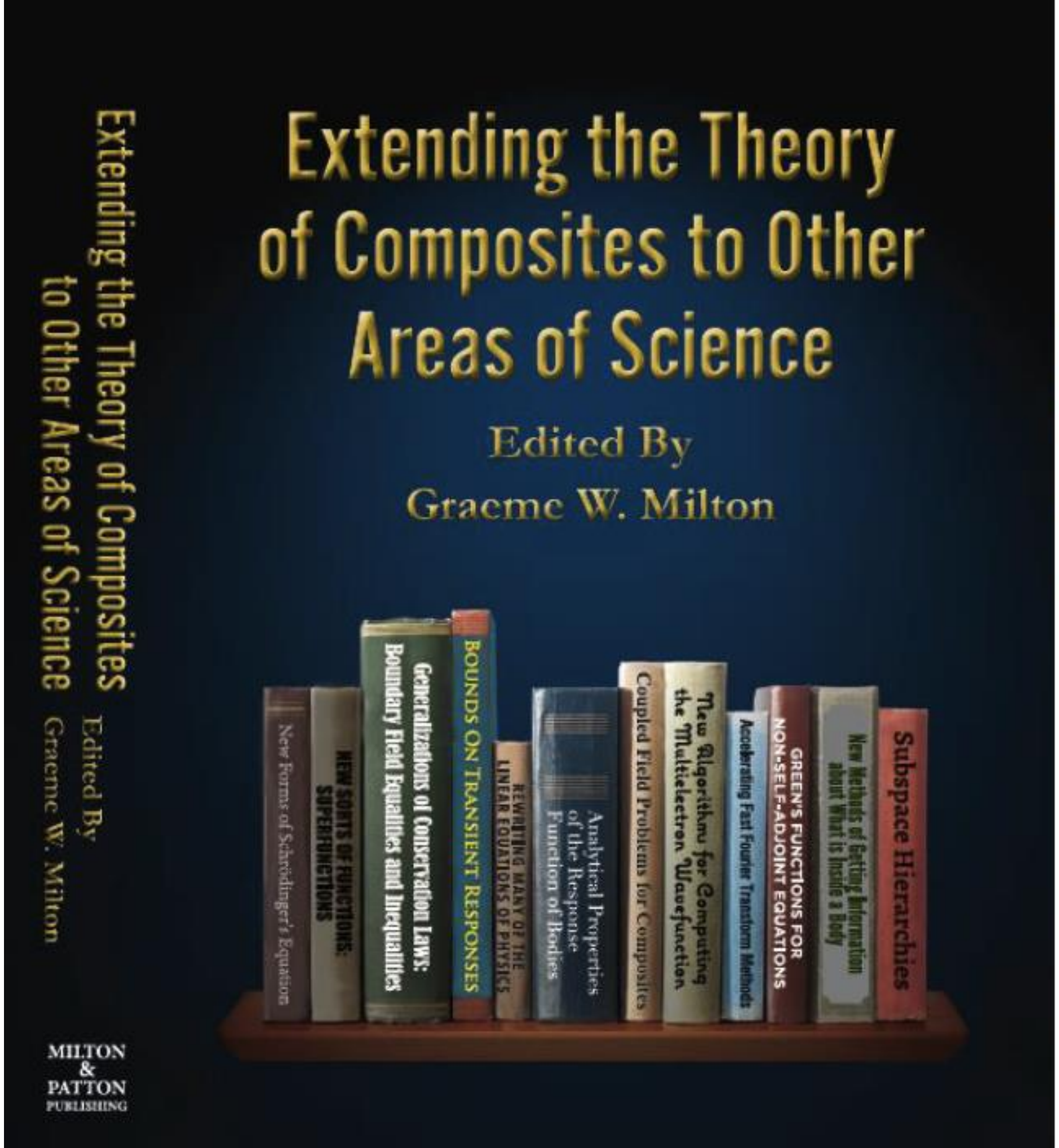


Dispersion diagrams for the microstructure with inclusions



Thank you for your attention!!

- Milton GW, Mattei O, 2017. *Field patterns: a new mathematical object.* Proc R Soc A. 473:20160819.
- Mattei O, Milton GW, 2017. *Field patterns without blow up.* To appear in New J Phys.
- Mattei O, Milton GW, 2017. *Field patterns: a new type of wave with infinitely degenerate band structure.* Submitted.



Chapters coauthored with:

Maxence Cassier
Ornella Mattei
Moti Milgrom
Aaron Welters

Available on my website

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