Contents

1 About this document .................................................. 1
2 Structure of the course .................................................. 1
  2.1 Students with disabilities ........................................... 1
  2.2 What you know already ............................................. 1
  2.3 Core material .................................................... 1
  2.4 The textbook ................................................... 1
  2.5 Grading .......................................................... 2
  2.6 Computer homework .............................................. 2
    2.6.1 Some help with Maple ....................................... 3
  2.7 Tutoring ........................................................ 4
  2.8 What we cover, and when and where ................................ 4
    2.8.1 Lectures: where and when ................................... 4
    2.8.2 Office hours: where and when ................................ 4
    2.8.3 Exams: where and when ..................................... 5
3 Monday/Wednesday (Section 1) Schedule .................................. 6
4 Tuesday/Thursday (Section 2) Schedule ................................... 8
5 Homework Assignments .................................................. 10
6 Why study PDE? .......................................................... 17
  6.1 What is a PDE? .................................................. 17
  6.2 But what do PDEs mean? ......................................... 18
  6.3 Typical problems ............................................... 19
  6.4 Example ....................................................... 19
7 Fourier series: waves in a box ........................................... 22
  7.1 Periodic functions .............................................. 22
  7.2 Basic vector geometry .......................................... 26
  7.3 Examples of inner products: sines and cosines ..................... 27
  7.4 Fourier series .................................................. 31
  7.5 Arbitrary periods ............................................... 36
  7.6 Texture ....................................................... 37
  7.7 Even and odd functions ....................................... 37
  7.8 Nonperiodic functions ......................................... 38
    7.8.1 Example: The square wave .................................. 40
  7.9 Parseval’s identity ............................................ 42
    7.9.1 Example: The square wave again ............................. 43
    7.9.2 Heisenberg’s uncertainty principle ......................... 44
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.7</td>
<td>Half lines</td>
<td>146</td>
</tr>
<tr>
<td>11.7.1</td>
<td>Derivatives</td>
<td>147</td>
</tr>
<tr>
<td>11.7.2</td>
<td>Heat in a half infinite wire</td>
<td>148</td>
</tr>
</tbody>
</table>
1  About this document

This document resides in

http://www.math.utah.edu/~mckay/3150.html

(the course web page). It provides the course notes for Math 3150: PDEs for Engineers, and will be developed as the course continues, providing some notes on the lectures; but the homework assignments given below will probably not change—if they do, you will be notified in class.

2  Structure of the course

2.1  Students with disabilities

The University of Utah seeks to provide equal access to its programs, services and activities for people with disabilities. If you will need accommodations in this class, reasonable prior notice needs to be given to the instructor and to the Center for Disability Services, 162 Olpin Union Building, 581-5020 (V/TDD) to make arrangements for accommodations.

All written information in this course can be made available in alternative format with prior notification.

2.2  What you know already

• 3 semesters of calculus, including vector or multivariable calculus

• Ordinary differential equations and linear algebra (as in the 2250 course, a 3 credit course with only a little linear algebra, mostly matrices, hopefully a touch of abstract vector spaces)

2.3  Core material

• Wave, heat and Laplace (=electrostatics) equations

• Changing coordinates and separating variables

• Fourier series and transforms

2.4  The textbook

The book is


which is available from the book store for $82 new and $65 used.

Previously, this course used

and


Chapters covered in the Gustafson–Wilcox book were 7–9 except for 7.5–7.7, 8.6, 9.6–9.7. Students who don’t like Asmar’s book could try looking at Gustafson–Wilcox as a supplementary source. (Math majors use Edwards and Penney.) But you will still need to have the Asmar book.

### 2.5 Grading

- Homework at least once a week
- 2 midterm tests (in class) and one final exam

<table>
<thead>
<tr>
<th>Task</th>
<th>Percentage of final grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homework</td>
<td>30%</td>
</tr>
<tr>
<td>Test #1</td>
<td>10%</td>
</tr>
<tr>
<td>Test #2</td>
<td>10%</td>
</tr>
<tr>
<td>Test #3</td>
<td>10%</td>
</tr>
<tr>
<td>Final exam</td>
<td>40%</td>
</tr>
</tbody>
</table>

No late homework will be accepted after the beginning of the class when they are due, without a request from the dean. Missed tests (or final exam) can not be rescheduled without a request from the dean.

### 2.6 Computer homework

Many homework problems will be carried out in Maple. There are computer labs in South Physics 205 and in EMCB (which most of you already know about). For more information, see

[www.math.utah.edu/ugrad/lab/](http://www.math.utah.edu/ugrad/lab/).

There is a short tutorial for Maple in

[www.math.utah.edu/~korevaar](http://www.math.utah.edu/~korevaar)

under the Math 2250 selection. Maple also provides extensive help and a new user’s tour. There are lots of examples of Maple programs on our course webpage.
2.6.1 Some help with Maple

1. There are computer labs in South Physics 205 and in EMCB. For more information, see

   www.math.utah.edu/ugrad/lab/.

2. To start Maple, type

   xmaple

3. To get help in Maple, look at the Help menu. Maple’s help facility gives examples. Cut and paste to try them out.

4. Maple commands must end with a semicolon

   ;

5. To get Maple to execute a command, you have to press return. Go to any line of the worksheet, and press return there, to have that line executed again.

6. Comments start with

   #

7. Maple distinguishes uppercase and lowercase letters; for example $\pi$ is written as Pi, not pi or PI.

8. Watch out for multiplication signs, written

   *

   You need to write them all the time. For instance,

   $2x$

   means nothing; you must type

   $2*x$

9. When Maple does something that doesn’t make sense to you, try the Edit menu, under Execute, and select Worksheet. This will get Maple to start all over from the beginning. You might also find that it helps to put a

   restart;

statement wherever you want to clear out all of the old variables, and at the beginning of the file.
10. If you have already used a variable for something else (like \( m \) in our case, which gets used quite a bit in \( b_m \) amplitudes), you need to wipe out its old values before using it again. For example, to wipe out the previous values of variables \( m \) and \( x \) type

\[
\begin{align*}
m & := 'm' ; \\
x & := 'x' ;
\end{align*}
\]

11. You can save your work into a file in several formats. Generally, use the default format (Maple Worksheet). Look in the File menu under Save As (or Save if you have previously saved the file).

12. To print, use the File menu Print command.

13. Maple has a tutorial, in the Help menu under New User’s Tour.

14. There is a short tutorial for Maple in

\[\text{www.math.utah.edu/~korevaar}\]

under the Math 2250 selection.

2.7 Tutoring

Free tutoring is available in Mines 210 (Mines is north of INSCC), available everyday, except weekends and holidays. Hours are posted on

\[\text{http://www.math.utah.edu/ugrad/tutoring.html}\]

Tutoring is also available through the University of Utah Tutoring Center, in the Student Services Building, room 330. Cost is $6.00 per hour. Students are given a list of tutors to contact and schedule a day, evening or weekend appointment. Low income students may qualify for free tutoring. For more information, call 581-5153 or visit \[\text{www.saff.utah.edu/Tutoring/}\].

2.8 What we cover, and when and where

2.8.1 Lectures: where and when

Section 1  Mondays and Wednesdays  10:45am–11:35am  OSH 107
Section 2  Tuesdays and Thursdays  9:40am–10:30am  BU C 301

2.8.2 Office hours: where and when

I will be in my office, which is JWB 126 (in the basement of the John Widtsoe Building, on President’s circle) 8:30am–9:30am, Monday–Thursday, and you can drop by then to ask questions; or you can schedule an appointment with me.
2.8.3 Exams: where and when

Section 1  Wednesday, May 8  10:30am–12:30pm  OSH 107
Section 2  Thursday, May 9  10:30am–12:30pm  BU C 301

The following schedules may be changed; if so you will be notified in class and updated versions will be posted.
# Monday/Wednesday (Section 1) Schedule

<table>
<thead>
<tr>
<th>Date</th>
<th>Topic</th>
<th>Textbook</th>
<th>Assignment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mon, Jan 7</td>
<td>Why learn PDE?</td>
<td>1.1–1.2, 3.1</td>
<td></td>
</tr>
<tr>
<td>Wed, Jan 9</td>
<td>Periodicity &amp; Fourier series</td>
<td>2.1–2.2</td>
<td></td>
</tr>
<tr>
<td>Mon, Jan 14</td>
<td>Playing with Fourier series</td>
<td>2.3–2.4</td>
<td>HW #1 &amp; #2 due</td>
</tr>
<tr>
<td>Wed, Jan 16</td>
<td>Energy &amp; Parseval's identity</td>
<td>2.5, 6.1</td>
<td>HW #3 due</td>
</tr>
<tr>
<td>Mon, Jan 21</td>
<td></td>
<td></td>
<td>Martin Luther King Day (no classes)</td>
</tr>
<tr>
<td>Wed, Jan 23</td>
<td>Complex Fourier series</td>
<td>6.1</td>
<td>HW #4 due</td>
</tr>
<tr>
<td>Mon, Jan 28</td>
<td>Oscillators</td>
<td>2.6</td>
<td>HW #5 due</td>
</tr>
<tr>
<td>Wed, Jan 30</td>
<td>Test #1</td>
<td>1.1–1.2, 2.1–2.5, 6.1</td>
<td>HW #6 due</td>
</tr>
<tr>
<td>Wed, Feb 27</td>
<td>Waves &amp; strings</td>
<td>3.2</td>
<td>HW #6 due</td>
</tr>
<tr>
<td>Mon, Mar 4</td>
<td>Separating variables</td>
<td>3.3</td>
<td>HW #7 due</td>
</tr>
<tr>
<td>Wed, Mar 6</td>
<td>d’Alembert’s method</td>
<td>3.4</td>
<td>HW #8 due</td>
</tr>
<tr>
<td>Mon, Mar 11</td>
<td>Heat</td>
<td>3.5</td>
<td>HW #9 due</td>
</tr>
<tr>
<td>Wed, Mar 13</td>
<td>Hot bars</td>
<td>3.6</td>
<td>HW #10 due</td>
</tr>
<tr>
<td>Mon, Mar 18</td>
<td>Heat &amp; waves in square plates</td>
<td>3.7–3.8</td>
<td>HW #11 due</td>
</tr>
<tr>
<td>Wed, Mar 20</td>
<td>Test #2</td>
<td>3.2–3.8</td>
<td></td>
</tr>
<tr>
<td>Mon, Mar 25</td>
<td>Changing coordinates</td>
<td>4.1–4.2</td>
<td>HW #12</td>
</tr>
<tr>
<td>Wed, Mar 27</td>
<td>Waves &amp; heat in disks</td>
<td>4.3–4.4</td>
<td>HW #13 due</td>
</tr>
<tr>
<td>Mon, Apr 1</td>
<td>Steady states of cylinders &amp; disks</td>
<td>4.5–4.6</td>
<td>HW #14</td>
</tr>
<tr>
<td>Wed, Apr 3</td>
<td>Bessel functions</td>
<td>4.7–4.8</td>
<td>HW #15 due</td>
</tr>
<tr>
<td>Date</td>
<td>Topic</td>
<td>Textbook</td>
<td>Assignment</td>
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<tr>
<td>Mon, Apr 8</td>
<td>Hanging chains</td>
<td>6.3</td>
<td>HW #16 due</td>
</tr>
<tr>
<td>Wed, Apr 10</td>
<td>Buckling beams</td>
<td>6.5</td>
<td>HW #17 due</td>
</tr>
<tr>
<td>Mon, Apr 15</td>
<td>More buckling beams</td>
<td>6.5</td>
<td>HW #18 due</td>
</tr>
<tr>
<td>Wed, Apr 17</td>
<td>Test #3</td>
<td>4.1–4.8,6.3,6.5</td>
<td></td>
</tr>
<tr>
<td>Mon, Apr 22</td>
<td>Fourier transforms</td>
<td>7.1–7.2</td>
<td>HW #19 due</td>
</tr>
<tr>
<td>Wed, Apr 24</td>
<td>Heat &amp; waves in infinite space</td>
<td>7.3</td>
<td>HW #20 due</td>
</tr>
<tr>
<td>Mon, Apr 29</td>
<td>Convolution</td>
<td>7.4</td>
<td>HW #21 due</td>
</tr>
<tr>
<td>Wed, May 1</td>
<td>The heat kernel</td>
<td>7.4</td>
<td>HW #22 due</td>
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<tr>
<td>Date</td>
<td>Topic</td>
<td>Textbook</td>
<td>Assignment</td>
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<tr>
<td>Thurs, Jan 3</td>
<td>Why learn PDE?</td>
<td>1.1–1.2, 3.1</td>
<td>HW #1 due</td>
</tr>
<tr>
<td>Tues, Jan 8</td>
<td>Periodicity &amp; Fourier series</td>
<td>2.1–2.2</td>
<td>HW #2 due</td>
</tr>
<tr>
<td>Thurs, Jan 10</td>
<td>Playing with Fourier series</td>
<td>2.3–2.4</td>
<td>HW #3 due</td>
</tr>
<tr>
<td>Tues, Jan 15</td>
<td>Energy &amp; Parseval’s identity</td>
<td>2.5, 6.1</td>
<td></td>
</tr>
<tr>
<td>Thurs, Jan 17</td>
<td>Complex Fourier series</td>
<td>6.1</td>
<td>HW #4 due</td>
</tr>
<tr>
<td>Tues, Jan 22</td>
<td>Oscillators</td>
<td>2.6</td>
<td>HW #5 due</td>
</tr>
<tr>
<td>Thurs, Jan 24</td>
<td>Test #1</td>
<td></td>
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<tr>
<td>Tues, Jan 29</td>
<td>Waves &amp; strings</td>
<td>3.2</td>
<td></td>
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<tr>
<td>Thurs, Feb 28</td>
<td>Separating variables</td>
<td>3.3</td>
<td></td>
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<tr>
<td>Tues, Mar 4</td>
<td>d’Alembert’s method</td>
<td>3.4</td>
<td></td>
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<tr>
<td>Thurs, Mar 7</td>
<td>Heat</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td>Tues, Mar 12</td>
<td>Heat &amp; waves in square plates</td>
<td>3.7–3.8</td>
<td></td>
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<td>Thurs, Mar 14</td>
<td>Test #2</td>
<td></td>
<td></td>
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<tr>
<td>Tues, Mar 19</td>
<td>Changing coordinates</td>
<td>4.1–4.2</td>
<td></td>
</tr>
<tr>
<td>Thurs, Mar 21</td>
<td>Waves &amp; heat in disks</td>
<td>4.3–4.4</td>
<td></td>
</tr>
<tr>
<td>Tues, Mar 26</td>
<td>Steady states of cylinders &amp; disks</td>
<td>4.5–4.6</td>
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<tr>
<td>Thurs, Apr 2</td>
<td>Bessel functions</td>
<td>4.7–4.8</td>
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<td>Thurs, Apr 4</td>
<td>Hanging chains</td>
<td>6.3</td>
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<td>Buckling beams</td>
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<tr>
<td>Tues, Apr 9</td>
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<td>Thurs, Apr 11</td>
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<td>Thurs, Apr 18</td>
<td>Heat &amp; waves in empty space</td>
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<tr>
<td>Tues, Apr 23</td>
<td>Convolution</td>
<td>7.4</td>
<td></td>
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<tr>
<td>Thurs, Apr 25</td>
<td>The heat kernel</td>
<td>7.4</td>
<td></td>
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<tr>
<td>Tues, Apr 30</td>
<td>The Poisson integral</td>
<td>7.5</td>
<td></td>
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<tr>
<td>Thurs, May 2</td>
<td>Cosine &amp; sine transforms, half lines</td>
<td>7.6–7.7</td>
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</table>

HW #18 due
HW #19 due
HW #20 due
HW #21 due
HW #22 due
HW #23 due
HW #24 due
5 Homework Assignments

Homework #1 Why learn PDE?
Due on Jan 14 for Monday/Wednesday section.
Due on Jan 8 for Tuesday/Thursday section.

1. Which functions satisfy which equations? (Show your calculations.)
   
   (a) \( u = \sin(x - ct) \) \hspace{1cm} (1) \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) (heat)
   
   (b) \( u = e^{-c^2 t} \sin(x) \) \hspace{1cm} (2) \( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \) (wave)
   
   (c) \( u = x - t \) \hspace{1cm} (3) \( \frac{\partial^2 u}{\partial x^2} = 0 \) (Laplace)

2. Show that the solutions \( u(x, y) \) to the PDE

\[
\frac{\partial u}{\partial y} - x \frac{\partial u}{\partial x} = 0
\]

are constant along all circles around the origin of coordinates. Hint: to move along a circle of radius \( r \), take

\[
x = r \cos \theta \\
y = r \sin \theta
\]

and think of \( r \) as a constant, and \( \theta \) as the variable. You will also need the chain rule for functions of several variables:

\[
\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]

3. The definition of derivative is

\[
\frac{dy}{dx} \sim \frac{y(x + \Delta x) - y(x)}{\Delta x}
\]

for small \( \Delta x \). (The ~ symbol means "is close to"). Use this to show that

\[
\frac{d^2 y}{dx^2} \sim \frac{y(x + \Delta x) - 2y(x) + y(x - \Delta x)}{(\Delta x)^2}.
\]

4. The heat equation

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}
\]
allows us to approximate solutions as follows: if we know \( u \) at time \( t \) and any location \( x \), then we can say that at time \( \Delta t \) later,
\[
u(x, t + \Delta t) \sim u(x, t) + \frac{\partial u}{\partial t} \Delta t.
\]
To make use of this, we also need to approximate \( \frac{\partial^2 u}{\partial x^2} \). This is easy:
\[
\frac{\partial^2 u}{\partial x^2} \sim \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2}.
\]
Now putting these two together, write an explicit formula to calculate \( u(x, t + \Delta t) \) at a later time \( t + \Delta t \) (a “time step”) from the values \( u(x, t) \), \( u(x + \Delta x, t) \) and \( u(x - \Delta x, t) \) (values at time \( t \)). You should get
\[
u(x, t + \Delta t) \sim u(x, t) + c^2 \Delta t \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2}
\]

5. Implement the ideas from the previous problem in a Maple worksheet, with two sequences, say \( u \) and \( u\text{Old} \). The values of \( u[i] \) should represent the value of \( u(x, t) \) at time \( t \) and position \( x \) where \( t = k\Delta t \), with \( k \) the number of time steps taken, and \( x = i/N \), where \( N \) is the number of samples of the function \( u(x, t) \) taken. Try \( N = 100 \) samples, \( \Delta t = \Delta x = 1/N \) and \( c = 1 \). To take a time step, copy the values from \( u \) into \( u\text{Old} \), so that these represent
\[
u\text{Old}[i] = u(x, t)
\]
with \( x = i/N \) and \( i = 1..N \) and then use the above results to fill \( u[i] \) with the numbers \( u(x, t + \Delta t) \) using the information stored in \( u\text{Old} \) to calculate those numbers. Start with \( u \) at time \( t = 0 \) being the function
\[
u(x, 0) = e^{-200(x - 1/2)^2}.
\]
Warning: do something reasonable about the ends: at \( i = 1 \) and \( i = N \), you can’t ask for the value of \( u\text{Old}[i-1] \) or \( u\text{Old}[i+1] \) since \( i \) runs from 1 to \( N \). Plot the function \( u(x, t) \) before taking any time steps, and then also plot the first five time steps. Hint: Maple is not very nice about plotting discrete data. I did my plots with
\[
\# \text{ Put stuff into a list } u
\]
\[
\text{plot}([\text{seq}([j, u[j]], j = 1..N)]);
\]
The plots should look like those in figure 1 on the following page. There is something terribly wrong: the actual solution of the heat equation with this initial temperature profile is shown in figure 2 on page 13. So this naive approach to solving partial differential equations, using barehanded calculus, gives very wrong answers. And we only tried to simulate the heat for 5/100 sec. Our simulation predicts hot spots of 400000\(^\circ\), when we only started with between 0\(^\circ\) and 1\(^\circ\) and no heat source.
Figure 1: A naive numerical approach to the heat equation
Figure 2: The exact solution of the heat equation

Homework #2 Periodicity and Fourier series
Due on Jan 14 for Monday/Wednesday section.
Due on Jan 10 for Tuesday/Thursday section.

1. Calculate
\[ \int_{\sqrt{2}+\sqrt{17}}^{2\pi+\sqrt{2}+\sqrt{17}} (1 + \sin x) \, dx \]
precisely (no decimal approximations, and show how to do it—don’t just write the answer).

2. Textbook page 20, #2,3,15, page 32 # 5–9: derive the Fourier series representations by hand, not by computer. Get Maple to graph the functions, and plot the partial sums.

Homework #3 Playing with Fourier series
Due on Jan 16 for Monday/Wednesday section.
Due on Jan 15 for Tuesday/Thursday section.

1. Textbook, page 41, #1–5: ignore the “points of discontinuity”—just derive the Fourier series.

2. Textbook, page 48, #1–5: plot the partial sums of 1,5 and 10 terms; you don’t have to comment on the graphs.

Homework #4 Energy and Parseval’s theorem
Due on Jan 23 for Monday/Wednesday section.
Due on Jan 17 for Tuesday/Thursday section.
1. Take a square wave, of height $H$ and width $W = 1/2$. Get Maple to graph the percentage of energy of the square wave contained in the first $N$ amplitudes: $a_0, \ldots, a_N$, as a function of $N$. Note that we have an exact expression for the total energy, and a sum to express the amount of energy in the first $N$ amplitudes. Get Maple to tell you how many amplitudes do we have to remember to capture 98% of the energy?

**Homework #5 Complex Fourier series**
Due on Jan 28 for Monday/Wednesday section.
Due on Jan 22 for Tuesday/Thursday section.

1. Textbook, page 61, #1,2,3,5.

**Homework #6 Oscillators**
Due on Feb 27 for Monday/Wednesday section.
Due on Jan 29 for Tuesday/Thursday section.

1. Calculate the first and second derivatives of the hyperbolic sine and cosine.

2. Show that the solutions of the harmonic oscillator equation for negative spring constant are given by hyperbolic sines and cosines, as mentioned in class.

3. How do you find the constants $a$ and $b$ appearing in the solutions of the harmonic oscillator equation in terms of position and velocity at time $t = 0$? Carry this out for positive, zero and negative spring constants.

4. Find the periodic solution of the forced harmonic oscillator equation with applied force

   $$F(t) = \cos \omega t.$$  

**Homework #7 Waves & strings**
Due on Mar 4 for Monday/Wednesday section.
Due on Jan 31 for Tuesday/Thursday section.

1. Textbook, section 3.2 #1,3,6,9.

**Homework #8 Separating variables**
Due on Mar 6 for Monday/Wednesday section.
Due on Feb 28 for Tuesday/Thursday section.

1. Textbook, section 3.3 #1,4,5 (do part (b) in Maple).

**Homework #9 d’Alembert’s method**
Due on Mar 11 for Monday/Wednesday section.
Due on Mar 5 for Tuesday/Thursday section.
1. Textbook, section 3.4 #1,2,8.

2. Write code in Maple to take functions \( f(x) \) (initial position) and \( g(x) \) (initial velocity) and a length \( L \) and

   (a) plot \( f(x) \)
   (b) plot an animation of the solution \( u(x,t) \) to the wave equation as on page 104 of the textbook.
   (c) Try out your animation on the initial conditions given in each of the problems 1,2,8 from section 3.4.
   (d) Print out the Maple worksheet showing the final state of the string at the end of each of these animations.

Homework #10 Heat
Due on Mar 13 for Monday/Wednesday section.
Due on Mar 7 for Tuesday/Thursday section.

   1. Textbook 3.5 #1,3,6,7.

Homework #11 Hot bars
Due on Mar 18 for Monday/Wednesday section.
Due on Mar 12 for Tuesday/Thursday section.

   1. Textbook 3.6 #1,3,7,16.

Homework #12 Heat & waves in square plates
Due on Mar 25 for Monday/Wednesday section.
Due on Mar 19 for Tuesday/Thursday section.

   1. Textbook 3.7 #1,6.
   2. Write Maple code to produce a 3D animation of the solution to textbook problem 3.7 #6.

Homework #13 Changing coordinates
Due on Mar 27 for Monday/Wednesday section.
Due on Mar 21 for Tuesday/Thursday section.

   1. Textbook 3.8 #1,2.
   2. Textbook 4.1 #1,3,9.

Homework #14 Waves & heat in disks
Due on Apr 1 for Monday/Wednesday section.
Due on Mar 26 for Tuesday/Thursday section.

   1. Textbook 4.2 #1,3,6.
2. Textbook 4.3 #1,2,3.

3. Write Maple code to animate the solution to textbook problem 4.3 #3.

Homework #15 Steady states of cylinders & disks
Due on Apr 3 for Monday/Wednesday section.
Due on Mar 28 for Tuesday/Thursday section.

1. Textbook 4.4 #2,3,7.
2. Textbook 4.5 #1,2,5.

Homework #16 Bessel functions
Due on Apr 8 for Monday/Wednesday section.
Due on Apr 2 for Tuesday/Thursday section.

1. Textbook 4.7 #3,6,
2. Textbook 4.8 #3,9,23,31.

Homework #17 Hanging chains
Due on Apr 10 for Monday/Wednesday section.
Due on Apr 4 for Tuesday/Thursday section.

1. Textbook 6.3 #1,4. Also get Maple to animate the chain for each of these examples.

Homework #18 Buckling beams
Due on Apr 15 for Monday/Wednesday section.
Due on Apr 9 for Tuesday/Thursday section.

1. Textbook 6.5 #2.
2. In Maple, create an animation of the vibrating beam described in textbook problem 6.5 #2.

Homework #19 More buckling beams
Due on Apr 22 for Monday/Wednesday section.
Due on Apr 16 for Tuesday/Thursday section.

1. Textbook 6.5 #4,6.
2. In Maple, create an animation of the vibrating beam described in textbook problems 6.5 #4,6.

Homework #20 Fourier transforms
Due on Apr 24 for Monday/Wednesday section.
Due on Apr 18 for Tuesday/Thursday section.
1. Textbook 7.1 #1,2,3,4,15.

2. Textbook 7.2 #1,2,3.

Homework #21 Heat & waves in infinite space
Due on Apr 29 for Monday/Wednesday section.
Due on Apr 23 for Tuesday/Thursday section.

1. Textbook 7.3 #1,2,20.

Homework #22 Convolution
Due on May 1 for Monday/Wednesday section.
Due on Apr 25 for Tuesday/Thursday section.

1. Textbook 7.2 #47,48,51,52.

Homework #23 The heat kernel
Due on Apr 30 for Tuesday/Thursday section.

1. Textbook 7.4 #2,3.

2. Create an animation in Maple to show the solution to textbook problem 7.4 #3.

3. Create an animation in Maple to simulate the solution to the heat equation with initial condition

\[ u(x,0) = e^{-200(x-1/2)^2}. \]

(Remember the first homework assignment you had: we tried to do this simulation without Fourier theory.)

Homework #24 The Poisson integral formula
Due on May 2 for Tuesday/Thursday section.

1. Textbook 7.5 #1,8,11.

6 Why study PDE?

6.1 What is a PDE?

PDE = Partial differential equation
Algebraically, what do they look like? Examples:

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \quad \text{(heat equation in the plane)}
\]

\[
\frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{(heat equation in the plane)}
\]

\[
0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad \text{(Laplace equation in space)}
\]

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u \frac{\partial u}{\partial x} \quad \text{(Burger’s equation)}
\]

This \( \frac{\partial u}{\partial x} \) means “derivative of \( u \) with respect to the \( x \) variable,” or “how fast \( u \) goes up when we tweek \( x \) up a little.”

Really the same thing as \( \frac{du}{dx} \) but we write \( \frac{\partial u}{\partial x} \) to remember that \( u \) may depend on other variables too. In one variable calculus, you used notation \( \frac{du}{dx} \).

This notation is used when \( u \) is independent of all variables except \( x \).

Consider the heat equation in the plane:

\[
\frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

Here \( u \) is temperature, \( x, y \) the variables of the plane, \( t \) is time, and \( c \) is a physical constant describing how quickly heat spreads through the material we are studying.

6.2 But what do PDEs mean?

Loosely:

\text{ODE } \sim \text{ a particle in classical mechanics}

\text{PDE } \sim \text{ a field (continuum spread out in space)}

\hspace{1cm} \text{such as fluid, heat, electrostatic energy,}

\hspace{1cm} \text{electromagnetic field, string, flexible surface,}

\hspace{1cm} \text{gas, electron cloud, etc.}

(This is not quite right: we will use ODE in this course for other purposes than just classical mechanics. And PDE can also be used for other things.)
Picture an ODE as telling you how a dot flies through space. A PDE tells you how a continuum object (string, fluid, heat, energy field, cloud, gravitational wave) flies through space (dynamics—such as heat flow, waves), or how it distributes itself while it sits still (statics—like static electricity, soap bubbles). But there is more too it than this—ODE are a bit more versatile. For example, mechanics of a rigid body (like a tennis racquet thrown in the air).

The point is that physical systems with finite number of degrees of freedom are described by ODEs, those with infinite degrees of freedom by PDEs. ODE and PDE describe all known physical phenomena.

6.3 Typical problems

In ODE problems, for example in classical mechanics, we know where the particle is and its velocity now, and we want to know where it will be later on. Note that the data of where it is now, and its velocity, consists of a few numbers—finitely many degrees of freedom.

In PDE dynamics problems, essentially the same: you known about what the field looks like now, and you ask the PDE what it will look like later. What it looks like now is called the initial condition. It is specified by writing out some functions, not just a few numbers—infinitely many degrees of freedom.

In statics problems, you measure the static field on the surface of an object, and the PDE tells you how the field is distributed inside. The field on the surface is called the boundary condition. Again, the boundary condition consists of functions, and any functions can occur there, so infinitely many degrees of freedom.

Solving a PDE is not like solving a bunch of ODEs, or solving ODEs in different variables. In general, it can’t be reduced to solving ODEs. But there are some special cases.

6.4 Example

The solutions of the ODE
\[
\frac{du}{dx} = 0
\]
are \( u(x) = c_0 \) where \( c_0 \) is any constant. Why? Because the equation says that the rate at which \( u \) varies when you vary \( x \) is 0, i.e. it doesn’t change.

The solutions of the PDE
\[
\frac{\partial u}{\partial x} = 0
\]
for a function \( u(x, y) \) are? The equation says that \( u \) doesn’t change when you vary \( x \), so it must be independent of \( x \), and so \( u(x, y) = u(y) \) is a function of \( y \) only. Any function will do.

Solved Problems #1
Figure 3: A solution of the heat equation in one variable: a lump of heat diffuses over time. Which direction is the space variable, and which is the time variable?
1. Which of the following functions

\[ u(x, t) = \frac{1}{t} e^{-\frac{x^2}{4t}} \]  \hspace{1cm} (a)

\[ u(x, t) = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \]  \hspace{1cm} (b)

\[ u(x, t) = e^{-\frac{x^2}{4t}} \]  \hspace{1cm} (c)

satisfies the heat equation on the line:

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \]

wherever \( t \neq 0 \)?

**Solution:** The answer is (b), which you find by taking derivative of \( u \) once in \( t \) to find the left hand side, and taking derivative of \( u \) twice in \( x \) to find the right hand side (then multiply by \( c^2 \)). □

2. Show that

\[ u(t, x) = \frac{x}{1-t} \]

satisfies

\[ \frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} \]

but that

\[ u(t, x) = c \frac{x}{1-t} \]

does not, for \( c \) a constant, except for certain values of \( c \). What are they?

**Solution:** Differentiate and you find

\[ \frac{\partial u}{\partial t} = \frac{cx}{(1-t)^2} \]

and

\[ u \frac{\partial u}{\partial x} = \frac{c^2 x}{(1-t)^2} \]

so that to have these equal for any \( x \) and \( t \) we must have \( c = 0 \) or \( c = 1 \).

□

To help you remember ODEs:

3. Change the ODE system

\[ \frac{dr}{dt} = r \cos^2 \theta \]

\[ \frac{d\theta}{dt} = -\cos \theta \sin \theta \]
from polar into rectangular \((x, y)\) coordinates, and find all of the solutions of it.

**Solution:** Write out

\[
x = r \cos \theta \\
y = r \sin \theta
\]

where \(r = r(t)\) and \(\theta = \theta(t)\) are varying in \(t\) according to the ODE system. Now differentiate both sides in \(t\), and you find (with a little trig)

\[
\frac{dx}{dt} = r \cos \theta = x \\
\frac{dy}{dt} = 0
\]

and you can easily integrate these equations to get

\[
x(t) = x_0 e^t \\
y(t) = y_0
\]

where \(x_0\) and \(y_0\) are arbitrary constants. □

## 7 Fourier series: waves in a box

Fourier series describe arbitrary sounds as combinations of “pure notes”, or arbitrary functions like the one in figure 4 as a sum of “pure notes” (physicists call them “modes”) like those in figures 5, 6 and 7.

Fourier series really do describe music; for example a synthesizer just makes a bunch of modes, and may sound like a violin or piano.

Why study Fourier series? For PDE problems: because heat, waves and electrostatics problems in a box are easy to solve for individual modes. Then we get all of the solutions by breaking functions down into modes. For signal analysis, because we can store information about repetitive patterns, like music, usually in a compact form. To get the pattern back, we just get some oscillators to fire up pure notes at the required frequencies.

### 7.1 Periodic functions

A periodic function looks like in figure 8: it just keeps repeating. Heart beats are nearly periodic.

We can draw vertical bars to indicate the periods, as in figure 9(a), or (for the same function) as in (b). The point is how far apart the bars are.
Figure 4: An arbitrary function

Figure 5: A mode
Figure 6: A low frequency mode

Figure 7: A high frequency mode
Figure 8: A periodic function

Figure 9: What counts are how far apart the bars are: the period
The width between these bars we call the period, and write it as $T$. Every $T$ units along, it looks just the same. In symbols:

$$f(x) = f(x + T)$$

Careful: a function which is periodic with period $T$ is also periodic with period $2T$, obviously. So when we say it has period $T$, we don’t rule out that it might have other periods too. Usually, we mean that $T$ is the smallest possible period, but sometimes we don’t.

Example: $\sin x = \sin(x + 2\pi)$. So $\sin$ has period $2\pi$.

Fact: If $f(x)$ is $T$ periodic, then adding up the values of $f$ over a whole period gives some number:

$$\int_a^{a+T} f(x) \, dx$$

which doesn’t dependent on which period we added up, i.e.

$$\int_a^{a+T} f(x) \, dx = \int_b^{b+T} f(x) \, dx$$

Why? Because $f(x)$ takes on the same values over any one period that it takes on over any other. And these integrals just add up all the values of $f(x)$.

Example:

$$\int_0^{2\pi} \sin x \, dx = \int_{-\pi}^{\pi} \sin x \, dx$$

Example: We have drawn the same function twice in figure 9. The integral between two neighboring vertical bars in (a) is the same as between two neighboring vertical bars as drawn in (b), because in each case they encompass one period.

### 7.2 Basic vector geometry

We said functions arising in PDE have infinitely many degrees of freedom. Vectors in three dimensional space have only three.

First vectors: you might have written them like

$$\vec{u} = (u_x, u_y, u_z)$$

with the $\rightarrow$ meaning that $\vec{u}$ is a vector. Or you might have written

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$$

but in either case, $\vec{u}$ is a vector with three components. Using the first notation, the dot product or inner product or scalar product is

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z;$$

a sum of products of $u$ values with $v$ values. We will call it the inner product.
Imitate this for functions: if $f(x), g(x)$ are functions, with period $T$, write

$$f \cdot g = \int f(x)g(x) \, dx$$

where the integral is carried out over a full period, i.e.

$$f \cdot g = \int_{a}^{a+T} f(x)g(x) \, dx$$

Again, a sum (=integral) of products of values of $f$ with those of $g$. But the traditional notation is a little different: we will write

$$(f, g)$$

instead of

$$f \cdot g$$

to avoid confusion with multiplication.

What does the inner product mean? For vectors, see figure 10.

What about inner product of functions? Like for vectors, it measures how similar two functions are. Like for vectors, it gives small values if you plug in small functions. Think of it as a measure of how strongly two functions agree; how in tune they are with each other musically. Small functions (near zero) are very weak, and can’t really sing very strongly. Taking inner product of a function $f$ with a function $g$ is measuring “how much” $g$ there is in $f$ and vice versa.

Warning: even though the triangular wave functions in figures 13 look alike, from the perspective of inner products, they are not closely related (i.e. their inner product is small), while the big triangular wave in figure 13 does have big inner product with the $\sin^4 x$ function.

### 7.3 Examples of inner products: sines and cosines

\[
\begin{align*}
\int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \begin{cases} 
0 & \text{if } m \neq n, \\
\pi & \text{if } m = n
\end{cases} \\
\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \begin{cases} 
0 & \text{if } m \neq n, \\
\pi & \text{if } m = n
\end{cases} \\
\int_{-\pi}^{\pi} \cos mx \sin nx \, dx &= 0
\end{align*}
\]

where $m, n$ are among $1, 2, 3, 4, \ldots$. These identities are all inner products. Note that if $m = 0$ we get $\sin mx = 0$ and $\cos mx = 1$ constant functions.

These functions

$$\cos mx, \sin mx$$

27
Figure 10: The meaning of the inner product
Figure 11: Similar functions: expect a big inner product

Figure 12: Very different functions—one bounces around much faster than the other. Expect small inner product
Figure 13: The first triangular wave has larger inner product with the $\sin^4 x$ than with the other triangular wave, because they bounce around the same way.
are going to be our “pure notes” or “modes”. (We will also take the constant function 1 to be a mode.) Note: they all have period $2\pi$. So if we knock a sine against a cosine, we get zero—they don’t have anything to do with each other. Careful: $\sin mx$ also has periods smaller than $2\pi$, with smallest period being $2\pi/m$. See figure 7.

When talking about sines and cosines we refer to their frequency as well as their period.

$$\text{frequency} = \frac{1}{\text{period}}$$

so the frequency of $\sin mx$ is

$$\text{freq of } \sin mx = \frac{m}{2\pi}$$

because the smallest period of $\sin mx$ is $2\pi/m$.

Important observation: even though $\sin x$ and $\cos x$ are just the same function shifted over, so they bounce around in much the same way, still their inner product is zero. This is because they are always just “out of phase” with each other: they don’t bounce together. Draw a picture if you don’t see what I mean—include several periods in your picture.

### 7.4 Fourier series

Fourier showed that every function $f(x)$ which has period $2\pi$ can be written as a sum of these sines and cosines, like

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x$$
$$+ a_2 \cos 2x + b_2 \sin 2x$$
$$+ a_3 \cos 3x + b_3 \sin 3x$$
$$+ \ldots$$

The $a_m$ coefficient measures how much $\cos mx$ appears in $f$. Call this a Fourier series.
Euler found formulas for these $a$’s and $b$’s:

\begin{align*}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx \\
a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x \, dx \\
b_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x \, dx \\
a_2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 2x \, dx \\
b_2 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin 2x \, dx \\
\vdots \\
a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx \\
b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx \\
\vdots
\end{align*}

These are inner products: to find out how much $\cos x$ there is in $f$ (the coefficient $a_1$) we knock a cosine against $f$ (the inner product). Then

$$a_1 = \frac{1}{\pi} (f(x), \cos x) \quad \text{(inner product)}$$

Be careful: there are also the annoying factors of $1/\pi$ in the front of the integrals—and of $1/2\pi$ in front of the $a_0$ integral. But these factors are just decoration; the core of these formulas are inner products.

We will call these $a$ and $b$ coefficients the amplitudes. Warning: some people call the numbers

$$\sqrt{a_m^2 + b_m^2}$$

the amplitudes, and call the $a_m$ and $b_m$ numbers the Fourier coefficients.

**Solved Problems #2**

1. Recall a trigonometry identity about

$$\cos^2 x + \sin^2 x$$

and use it to calculate

$$\int_{-\pi}^{\pi} (\cos^2 x + \sin^2 x) \, dx.$$

Now show (again by trig) that $\cos^2 x$ and $\sin^2 x$ are related by

$$\cos^2 x = \sin^2(x + c)$$
for some $c$. What is $c$? (Hint: draw a picture of both functions. This $c$
represents how you have to shift the graphs around to get one function to
look like the other.)

From our work on integrals of periodic functions, why is

$$\int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \sin^2 x \, dx ?$$

Put this last equation together with the result you got for

$$\int_{-\pi}^{\pi} (\cos^2 x + \sin^2 x) \, dx .$$

to find

$$\int_{-\pi}^{\pi} \cos^2 x \, dx \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2 x \, dx$$

**Solution:**

$$\cos^2 x + \sin^2 x = 1$$

so that

$$\int_{-\pi}^{\pi} (\cos^2 x + \sin^2 x) \, dx = 2\pi .$$

We also know that

$$\cos x = \sin(x + \pi/2)$$

so that

$$\int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \sin^2(x + \pi/2) \, dx$$

$$= \int_{-\pi}^{\pi} \sin^2 x \, dx$$

(we are just shifting the cosine over, and integrating over a full period).
The integral of a periodic function over a period is independent of which
period. Therefore

$$\int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \sin^2 x \, dx .$$

But the sum of these two integrals is $2\pi$, so each of them must be $\pi$:

$$\int_{-\pi}^{\pi} \cos^2 x \, dx = \int_{-\pi}^{\pi} \sin^2 x \, dx = \pi$$

□
2. Now use the last result to find

\[ \int_{-\pi}^{\pi} \cos^2 mx \, dx \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2 mx \, dx \]

for \( m = 1, 2, 3, 4, \ldots \). Hint: stretch or squish the \( x \) variable to get back to the previous problem (substituting a new variable in the integral).

**Solution:** Let \( u = mx \), so \( x = u/m \). Watch the limits of integration very carefully:

\[
\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \cos^2 \frac{u}{m} \, \frac{du}{m} \quad \text{Note } dx = du/m \text{ and limits}
\]

\[= \frac{1}{m} \int_{-\pi}^{\pi} \cos^2 u \, du\]

\[= \frac{1}{m} \int_{-\pi}^{\pi} \cos^2 u \, du \quad \text{Sum of } m \text{ periods}
\]

\[= \int_{-\pi}^{\pi} \cos^2 u \, du\]

\[= \pi\]

\[\square\]

3. Use the trig identity

\[\cos(A + B) = \cos A \cos B - \sin A \sin B\]

to show that

\[\cos mx \cos nx = \frac{1}{2} \left( \cos((m + n)x) + \cos((m - n)x) \right)\]

Use this to carry out the integral

\[\int_{-\pi}^{\pi} \cos mx \cos nx \, dx\]

when \( m \neq n \).

**Solution:**

\[\cos(m + n)x = \cos(mx + nx)\]

\[= \cos mx \cos nx - \sin mx \sin nx\]

\[\cos(m - n)x = \cos(mx + (-nx))\]

\[= \cos mx \cos(-nx) - \sin mx \sin(-nx)\]

\[= \cos mx \cos nx + \sin mx \sin nx\]
(because sin is odd, and cos is even). Adding these together
\[ \cos(m + n)x + \cos(m - n)x = 2 \cos mx \cos nx \]

Taking the integral:
\[ \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(m + n)x + \frac{1}{2} \int_{-\pi}^{\pi} \cos(m - n)x \]

We can calculate that for \( \alpha \neq 0 \),
\[ \int \cos \alpha x \, dx = \frac{\sin \alpha x}{\alpha} \]
so that these integrals become
\[ \frac{1}{2} \sin(m + n)x \bigg|_{x=\pi}^{x=-\pi} = 0 \]
and
\[ \int_{-\pi}^{\pi} \cos(m - n)x \, dx = \begin{cases} \sin(m-n)x \bigg|_{x=\pi}^{x=-\pi} & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases} \]
\[ = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases} \]
Therefore
\[ \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq 0 \end{cases} \]
\[ \square \]

4. Writing a function as a Fourier series
\[ f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \ldots \]
prove the equation (of Euler)
\[ a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x \, dx \]
by plugging the Fourier series into the integral on the right hand side and expanding it out. You will need the result of the last exercise, together
with the equations
\[ \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \]
\[ \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]

(All the other equations Euler found for Fourier coefficients work out the same way.)

**Solution:** Plug in:

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots) \cos x \, dx \\
= \frac{a_0}{\pi} \int_{-\pi}^{\pi} \cos x \, dx + \frac{a_1}{\pi} \int_{-\pi}^{\pi} \cos^2 x \, dx + \frac{b_1}{\pi} \int_{-\pi}^{\pi} \cos x \sin x \, dx + \ldots \\
= \frac{a_0}{\pi} 0 + \frac{a_1}{\pi} \pi + \frac{b_1}{\pi} 0 + \frac{a_2}{\pi} 0 + \ldots \\
= a_1
\]

\[
\square
\]

### 7.5 Arbitrary periods

So far we only have Fourier series for functions which are 2\(\pi\) periodic. For functions with period \(T\), we just rescale the \(x\) axis to get period 2\(\pi\), by plugging in \(\frac{2\pi}{T} x\) instead of \(x\). So by just this rescaling, we get

\[
f(x) = a_0 + a_1 \cos \left(\frac{2\pi}{T} x\right) + b_1 \sin \left(\frac{2\pi}{T} x\right) \\
+ a_2 \cos \left(\frac{4\pi}{T} x\right) + b_2 \sin \left(\frac{4\pi}{T} x\right) \\
+ a_3 \cos \left(\frac{6\pi}{T} x\right) + b_3 \sin \left(\frac{6\pi}{T} x\right) \\
+ \ldots \\
+ a_m \cos \left(\frac{2\pi m}{T} x\right) + b_m \sin \left(\frac{2\pi m}{T} x\right) \\
+ \ldots
\]
and the equations for the Fourier series are

\[ a_0 = \frac{1}{T} \int_{0}^{T} f(x) \, dx \]

\[ a_m = \frac{2}{T} \int_{0}^{T} f(x) \cos \left( \frac{2\pi m}{T} x \right) \, dx \]

\[ b_m = \frac{2}{T} \int_{0}^{T} f(x) \sin \left( \frac{2\pi m}{T} x \right) \, dx \]

Again the idea is that we knock \( f \) against a cosine or sine of the required frequency, calculating the inner product, and (besides having the factors of \( 2/T \), which are not interesting) we find out exactly how much of that frequency cosine or sine lives in \( f \).

### 7.6 Texture

Compare the two functions in figure 14. Texture is controlled by high frequency terms (i.e. \( \cos mx, \sin mx \) with large \( m \)) while general (large scale) shape is controlled with low frequency stuff. We will get a clearly idea of this later on.

### 7.7 Even and odd functions
A function \( f(x) \) is even if
\[
f(x) = f(-x)
\]
(like \( \cos x \) or \( x^2 \)). In pictures, it is symmetric across the \( y \) axis.

A function \( f(x) \) is odd if
\[
f(x) = -f(-x)
\]
(like \( \sin x \) or \( x^3 \)). In pictures, a bump going up for a positive \( x \) has a corresponding trough going down for \( -x \).

Every function splits into even and odd part:
\[
f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}
\]

Since \( \cos mx \) is even for all \( m \), and \( \sin mx \) is odd, we guess correctly that odd functions will have only sine terms in their Fourier series and even functions have only cosines.

### 7.8 Nonperiodic functions

Start with a function, \( f(x) \), and cut out an interval of where it is defined. Take it to be \( 0 \leq x \leq L \), say, for some number \( L \), as in figure 16 (a). Now how can we put together a periodic function out of this piece? First, as in figure 16 (b), we just flip it over the \( y \) axis to get it to be even. Then we cut and paste these blocks to build a periodic function, as in figure 16 (c). Call this function \( g(x) \).
(a) Start with a function defined on $0 \leq x \leq L$

(b) Flip it over to get an even function

(c) Make copies of it so it is periodic

Figure 16: Building an even periodic function out of any function
The resulting function will be even and periodic—the sine components of its Fourier series must vanish. So it is all cosines:

\[ g(x) = a_0 + a_1 \cos \left( \frac{2\pi}{T} x \right) + \ldots \]

Here the period \( T \) is \( T = 2L \) (look at the picture). Because the integrals of an even \( 2L \) periodic function are the same over \( 0 \leq x \leq L \) as over \( -L \leq x \leq 0 \), we find the cosine terms are

\[
an_m = \frac{2}{T} \int_{-L}^{L} g(x) \cos \left( \frac{2\pi m}{T} x \right) dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{\pi m}{L} x \right) dx
\]

except for \( a_0 \), which is still the average:

\[
a_0 = \frac{1}{L} \int_{0}^{L} f(x) dx
\]

Why not try making a periodic odd function out of \( f(x) \) instead? You see the problem in figure [17]. Unless \( f(x) \) vanishes at the two ends of the interval, it won’t patch together nicely: there will be jumps. Jumps are not always bad, but the original function varied continuously, without jumps. Jumps require very high frequencies. So spurious jumps mess with the high frequency Fourier components. This is called “ringing”.

Example: When Korean Airlines Flight 007 was blown up by a Soviet fighter plane in Soviet air space, the Soviets claimed it was spying. The United States recovered the flight recorder, and it passed through the CIA first and then to the FAA. The FAA looked at Fourier series of the flight recorder data (they usually do), and found ringing. This was because someone had cut a bit out of the tape, and pasted the tape back together.

7.8.1 Example: The square wave

Start with a function \( f(x) \) defined for \( 0 \leq x \leq 1 \) by

\[
f(x) = \begin{cases} H & 0 \leq x \leq W \\ 0 & W \leq x \leq 1 \end{cases}
\]

(here \( W, H \) are some constants) and extend it to an even function. This is called a square wave (see figure [18]). The amplitudes are

\[
a_0 = \int_{0}^{1} f(x) dx = HW
\]
(a) Some function we want to analyse with Fourier series

(b) Flipping it over to make an odd function

(c) Several periods of the odd periodic function

Figure 17: The odd periodic function we end up with has sharp jumps, unless we start with a function which vanishes at $x = 0, L$ (the end points)
Figure 18: A square wave of height $H = 4$, width $W = 1/4$

and

$$a_m = 2 \int_0^1 f(x) \cos(\pi mx) \, dx$$

$$= 2 \int_0^W H \cos(\pi mx) \, dx$$

$$= 2H \int_0^W \frac{d}{dx} \left( \frac{1}{\pi m} \sin(\pi mx) \right) \, dx$$

because $\frac{d}{dx} \sin x = \cos x$

$$= 2H \frac{1}{\pi m} \sin(\pi mx) \bigg|_0^W$$

$$= 2H \frac{1}{\pi m} \sin(\pi mW)$$

because $\sin 0 = 0$

How does this look? Since we know $-1 \leq \sin x \leq 1$ for any $x$, we find these $a_m$ are decaying away at a rate of $1/m$ at least.

### 7.9 Parseval’s identity

The length of a vector is

$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

(Pythagorean theorem).

By (loose) analogy with classical mechanics call

$$\|\vec{u}\|^2 = u_x^2 + u_y^2 + u_z^2$$

the energy.
We have an inner product for periodic functions; what is the Pythagorean theorem? Define
\[ \|f\| = \sqrt{(f, f)} \]
so that the energy of a function is
\[ \|f\|^2 = \int_{\text{period}} f(x)^2 \, dx \]
as for vectors. We can use this to calculate the energy. The analogue of the Pythagorean theorem is then
\[ \|f\|^2 = T a_0^2 + \frac{T}{2} (a_1^2 + b_1^2 + a_2^2 + b_2^2 + \ldots) \]
called Parseval’s identity. It says that the total energy of a function is the sum of the energy living in each frequency. We can use it to find out how much of the total energy of the function lives in each frequency.

7.9.1 Example: The square wave again
Suppose the square wave is very skinny (small \( W \)), but very tall (big \( H \)). More precisely, let us take \( H \sim 1/W \). How big are the amplitudes? For amplitudes \( a_m \) with \( m \ll 1/W \) (i.e. the first bunch of them—small \( W \) means this is a lot of amplitudes), we find
\[ a_m = \frac{2H}{\pi m} \sin(\pi m W) \]
\[ \sim \frac{2H}{\pi m} \pi m W \]
because \( \sin x \sim x \) if \( x \) is small
\[ = 2HW \]
\[ \sim 2 \]
So the first long bunch of amplitudes each have energy about \( 2^2 = 4 \). The total energy is
\[ 2 \int_0^1 f(x)^2 \, dx = 2 \int_0^W H^2 \, dx \]
\[ = 2H^2 W \]
\[ \sim \frac{2}{W} \]
(since \( H \sim 1/W \)).

Summing up, the total energy is about \( 2/W \) while each of the first bunch of amplitudes has energy about 4, and the amplitudes \( a_m \) far out away from those first few (say, \( m \gg 1/W \)) are
\[ a_m = \frac{2H}{\pi m} \sin(\pi m W) \]
Figure 19: The amplitudes $a_m$ for a square wave with $W = 0.1$ and with $W = 0.01$. Notice how fast they decay for $W = 0.1$, while they take longer to start decaying, and stay near $a_m = 2$ longer, for $W = 0.01$

no bigger in energy than about

$$a_m^2 = \frac{4H^2}{\pi^2 m^2} \sin^2(\pi m W) \lesssim \frac{4}{\pi^2 m^2 W^2}$$

so decaying like $1/m^2$. See figure 19

7.9.2 Heisenberg's uncertainty principle

Roughly, a small high bump $f(x)$ has energy spread out through many amplitudes. In our square wave, a small sharp blip of a square wave spreads its energy evenly through the first bunch of amplitudes.

For a computer, this means you would have to store a lot of amplitudes.

7.10 Complex Fourier series

Sines and cosines are ugly algebraically. The algebra of complex numbers simplifies them, and thereby simplifies Fourier series.

7.10.1 Review of complex numbers

Let

$$i = \sqrt{-1} \text{ or } j = \sqrt{-1}$$

You can use either one. Math/Physics people like $i$, engineers (to avoid confusion with $I$ as current) use $j$. Once you know

$$i^2 = -1$$
you know it all—you can add, subtract, multiply and divide. Then you can define other functions, like \( \sin, \cos, \) etc. on complex numbers by power series expansions.

Complex numbers \( z = x + iy \) are vectors in the plane, with \( (x, y) \) components. (Draw it.) Complex conjugation

\[
z = x + iy \mapsto \bar{z} = x - iy
\]

is reflection in the \( x \) axis. Some people prefer to write

\[
z^* \text{ or } z^\dagger
\]

for complex conjugate. We can write any complex number in polar coordinates as

\[
z = re^{i\theta}
\]

where \( r = \|z\| \) and \( \theta \) is the angle of the \( z \) vector from the \( x \) axis. (Draw it.) In these polar coordinates, complex conjugation is

\[
\bar{z} = re^{-i\theta}.
\]

You should know

\[
e^{ix} = \cos x + i\sin x
\]
for any real number (or complex number) x. From this you can get

\[
\cos x = \frac{e^{ix} + e^{-ix}}{2}, \\
\sin x = \frac{e^{ix} - e^{-ix}}{2i}.
\]

Also, plugging in \(x = 2\pi k\) for \(k\) an integer,

\[
e^{2\pi ik} = \cos (2\pi k) + i \sin (2\pi k) = 1
\]

Important fact: working from either \(z = x + iy\) or from \(z = re^{i\theta}\), you can easily calculate

\[
z\bar{z} = x^2 + y^2 = r^2
\]

so \(z\bar{z}\) is the squared length of the vector \(z = (x, y)\).

Complex vectors

\[
\vec{u} = (u_1, u_2, u_3)
\]

(with \(u_1, u_2, u_3\) complex numbers) use a fancier inner product:

\[
(\vec{u}, \vec{v}) = u_1\bar{v_1} + u_2\bar{v_2} + u_3\bar{v_3}
\]

For real vectors, this is just the usual inner product. Note that

\[
(\vec{u}, \vec{u}) = u_1\bar{u_1} + u_2\bar{u_2} + u_3\bar{u_3}
\]

is the squared length of the vector \(\vec{u}\).

For complex functions with period \(T\), say \(f(x), g(x)\) (a real variable \(x\)), we define the inner product

\[
(f, g) = \int_0^T f(x) \overline{g(x)} \, dx
\]

to make it look like vectors. Warning: physicists write

\[
(f|g) = \int_0^T \overline{f(x)} g(x) \, dx
\]

### 7.10.2 Application to Fourier series

So far, complex numbers have been just more definitions. But from

\[
e^{ix} = \cos x + i \sin x
\]

we see that cosines and sines are just real and imaginary parts of this exponential type function. So everything can be written using these exponentials: any
function $f(x)$ of period $T$ (real or complex valued, but a function of a real variable $x$) can be written

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx/T}$$

Why? Just use the algebraic equations above for cosine and sine in terms of $e^{ix}$ and $e^{-ix}$, and play with the algebra.

Inner products:

$$\int_0^T e^{2\pi imx/T} e^{-2\pi inx/T} \, dx = \begin{cases} T & \text{when } m = n \\ 0 & \text{otherwise} \end{cases}$$

Look at this as the inner product of

$$e^{2\pi imx/T}$$

with

$$e^{2\pi inx/T}$$

Watch out for the signs in front of $m, n$!

Proof. First we need to remember the integral

$$\int e^{\alpha x} \, dx = \frac{e^{\alpha x}}{\alpha}$$

for $\alpha$ a constant. Then

$$\int_0^T e^{2\pi imx/T} e^{-2\pi inx/T} \, dx = \int_0^T e^{2\pi i(m-n)x/T} \, dx$$

so if $m = n$, this is

$$\int_0^T e^0 \, dx = \int_0^T 1 \, dx = T$$

But if $m \neq n$, then

$$\int_0^T e^{2\pi i(m-n)x/T} \, dx = \left. \frac{e^{2\pi i(m-n)x/T}}{2\pi i(m-n)/T} \right|_0^T - \frac{e^0}{2\pi i(m-n)/T}$$

$$= \frac{1}{2\pi i(m-n)/T} - \frac{1}{2\pi i(m-n)/T}$$

$$= 0$$
Warning: these $c_k$ are complex numbers, even if $f(x)$ is a real valued function.

To get the Fourier coefficients $c_k$:

$$c_k = \frac{1}{T} \int_0^T f(x) e^{-2\pi ikx/T} \, dx$$

which is also an inner product. We no longer have a weird special case formula for $a_0$ to worry about, and there is only one kind of waveform, not two (sines and cosines).

The proof is easy:

$$\frac{1}{T} \int_0^T f(x)e^{-2\pi ikx/T} \, dx = \frac{1}{T} \int_0^T \left( \sum_{m=-\infty}^{m=\infty} c_m e^{2\pi imx/T} \right) e^{-2\pi ikx/T} \, dx$$

$$= \frac{1}{T} \sum_{m=-\infty}^{m=\infty} c_m \int_0^T e^{2\pi mx/T} e^{-2\pi kx/T} \, dx$$

$$= \frac{1}{T} \sum_{m=-\infty}^{m=\infty} c_m \begin{cases} T & \text{if } m = k \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{T} c_k T$$

$$= c_k$$

Summing up:

$$c_k = \frac{1}{T} \int_0^T f(x)e^{-2\pi ikx/T} \, dx$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx/T}$$

7.10.3 Example: differentiating

We know

$$\frac{d}{dx} e^x = e^x$$

and therefore

$$\frac{d}{dx} e^{\alpha x} = \alpha e^{\alpha x}$$

if $\alpha$ is a constant. Take a complex Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx/T}$$

and differentiate:

$$\frac{df}{dx} = \sum_{k=-\infty}^{\infty} c_k \frac{2\pi ik}{T} e^{2\pi ikx/T}$$
so the amplitudes of $df/dx$ are

$$\frac{2\pi ik}{T} c_k$$

where $c_k$ are the amplitudes of $f(x)$.

<table>
<thead>
<tr>
<th>Function</th>
<th>Amplitudes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>$c_k$</td>
</tr>
<tr>
<td>$df/dx$</td>
<td>$\frac{2\pi ik}{T} c_k$</td>
</tr>
<tr>
<td>$d^2 f/dx^2$</td>
<td>$(\frac{2\pi ik}{T})^2 c_k$</td>
</tr>
<tr>
<td>$d^3 f/dx^3$</td>
<td>$(\frac{2\pi ik}{T})^3 c_k$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d^N f/dx^N$</td>
<td>$(\frac{2\pi ik}{T})^N c_k$</td>
</tr>
</tbody>
</table>

7.10.4 Example: periodic ODE problems

Find all solutions of the ODE

$$f''(x) = -f(x)$$

(the harmonic oscillator)

which are periodic with period $2\pi$.

**Solution:** Write the function as a Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} .$$

The amplitudes of $f''(x)$ must be

$$(ik)^2 c_k = -k^2 c_k .$$

For this to match the Fourier series of $-f(x)$, we need all the amplitudes to match:

$$-k^2 c_k = -c_k$$

or, if $c_k \neq 0$, we divide by $c_k$ to find

$$k^2 = 1$$

so that

$$k = \pm 1 .$$

So the answer has to have Fourier series

$$f(x) = c_{-1} e^{-ix} + c_1 e^{ix} .$$

Check by differentiating that this works. So these are all the solutions, where $c_{-1}$ and $c_1$ are arbitrary complex numbers. Important: which ones are the real valued solutions? □
7.10.5 Parseval's theorem

For \( f(x) \) real or complex valued, with period \( T \)

\[
\text{energy}(f) = \|f\|^2 = (f,f) = \int_0^T |f|^2 = T \sum_k |c_k|^2
\]

and the proof is easy: just plug in the Fourier series for \( f(x) \) into the integral, and it falls out.

Solved Problems #3

1. Find all of the solutions of the harmonic oscillator equation

\[ f''(x) = -f(x) \]

which are periodic of period \( T \). What values of \( T \) are allowed? What relations on the amplitudes \( c_k \) do you need to force the solutions \( f(x) \) to be real valued functions? Write all of the real valued solutions \( f(x) \) in terms of sines and cosines.

Solution: Write

\[ f(x) = \sum_k c_k e^{2\pi i k x/T} \]

and differentiate twice, to find

\[ f''(x) = \sum_k \left( \frac{2\pi i k}{T} \right)^2 c_k e^{2\pi i k x/T} \]

so that to have \( f''(x) = -f(x) \)

\[ \left( \frac{2\pi i k}{T} \right)^2 c_k = -c_k \]

for each integer \( k \). Either \( c_k = 0 \), i.e. that amplitude vanishes, or else

\[ \left( \frac{2\pi i k}{T} \right)^2 = -1 \]

or

\[ k = \pm \frac{T}{2\pi} . \]

From this we learn two things: that \( T \) is a \( 2\pi \) multiple of some integer, say

\[ T = 2\pi N \]

for \( N \) some positive integer. It has to be positive, because \( T \) is a period, so must be positive (distance between the places where the value of the function repeats). Then also we learn that \( k = \pm N \), so that our function looks like

\[ f(x) = c_{-N} e^{2\pi i (-N)x/T} + c_N e^{2\pi i N x/T} \]
where $c_{-N}$ and $c_N$, the amplitudes of $f(x)$, are any complex numbers. Plugging in $T = 2\pi N$ gives
\[ f(x) = c_{-N}e^{-ix} + c_Ne^{ix} \]
which shows that $f(x)$ has period $2\pi$, not merely $2\pi N$. In fact, this is the solution obtained in class.

To make $f(x)$ real, we need $f(x) = \overline{f(x)}$ for any $x$. This gives
\[ c_{-N}e^{-ix} + c_Ne^{ix} = \overline{c_{-N}e^{ix}} + \overline{c_Ne^{-ix}} \]
or
\[ c_{-N} = \overline{c_N} \]
so that
\[ f(x) = \overline{c_N}e^{-ix} + c_Ne^{ix} \]
Write $c_N$ as $c = a + ib$ for simplicity. Then
\[ f(x) = (a - ib)e^{-ix} + (a + ib)e^{ix} \]
which expands out, using
\[ e^{ix} = \cos x + i\sin x \]
into
\[
\begin{align*}
  f(x) &= \left(a \cos x - b \sin x - ia \sin x - ib \cos x\right) + \left(a \cos x - b \sin x + ia \sin x + ib \cos x\right) \\
  &= 2a \cos x - 2b \sin x
\end{align*}
\]
These $a$ and $b$ are arbitrary real numbers, which we can see by plugging this into the differential equation, and seeing that it is a solution, no matter what $a$ and $b$ are. □

\section*{7.11 The harmonic oscillator}
A spring (a.k.a. a harmonic oscillator) satisfies a simple law: the force pulling the spring is proportional to the amount of stretch. Suppose that the spring sits at $x = 0$ when you don’t stretch it. Then the force is $F = -kx$, for a positive number $k$. Using Newton’s law $F = ma$ we get
\[ m \frac{d^2x}{dt^2} = -kx. \]
Plugging in
\[ x(t) = a \cos \omega_0 t + b \sin \omega_0 t \]
where \(a\) and \(b\) are any constants, we find that this function satisfies the oscillator equation, precisely if the frequency \(\omega_0\) is
\[ \omega_0 = \sqrt{\frac{k}{m}}. \]

The period is
\[ T_0 = \frac{2\pi}{\omega_0}. \]

But in fact, we know that there is only a 2 parameter family of solutions, because it is a 2nd order equation (a \(k\)-th order equation has a \(k\) parameter family of solutions). Since we have two free parameters, \(a\) and \(b\), we can believe that these are all of the solutions.

In the limit as the spring exerts less force, \(k \to 0\), we have a very long period before the spring repeats. If we take \(k = 0\), then there are no periodic solutions, and the solutions are
\[ x(t) = a + bt \]
just flying off linearly into space.

If we take the unphysical (but useful) step of allowing negative spring constant \(k\), so that the spring actually pushes you away with greater force the further you are from equilibrium, then the solutions are
\[ x(t) = a \cosh(\omega_0 t) + b \sinh(\omega_0 t) \]
where
\[ \cosh(t) = \frac{e^t + e^{-t}}{2}, \]
\[ \sinh(t) = \frac{e^t - e^{-t}}{2} \]
(called the hyperbolic trigonometric functions) and we write
\[ \omega_0 = \sqrt{\frac{-k}{m}} \]
(which is not really anything like a frequency). These solutions fly off away from equilibrium at exponentially growing speed. But they will be useful to us when we study electrostatics.

Crucial idea: if the spring constant is positive, then there are lots of zeros—the spring returns to equilibrium over and over; if the spring constant vanishes, or is negative, then there is either one zero or none.

Back to the physical problem: positive spring constant. Since the solutions of the harmonic oscillator are sines and cosines, we can view Fourier series as breaking up functions into pure oscillators—picture the function being the
(a) Positive spring constant  
(b) Zero spring constant  
(c) Negative spring constant

Figure 21: Varying the spring constant in the harmonic oscillator equation
height of a spring, but the spring is not just a simple spring, it is a collection of springs of various sizes, each oscillating with its own spring constant, and all of them glued together end to end. Such a device can have arbitrarily complicated oscillations.

Consider a forced oscillator:

$$\frac{dx^2}{dt^2} + \frac{k}{m}x = \frac{F(t)}{m}$$

where $F(t)$ is the external force applied to the oscillator. We will assume that the applied force is periodic. It is convenient to write $k/m = \omega_0^2$. Let us once again look for periodic solutions, but now take the period to be $T$, the period of the force, not the period $T_0$ of the unforced oscillator. Let $\omega = 2\pi/T$. Taking complex Fourier series

$$x(t) = \sum_n a_n e^{in\omega t}$$

$$F(t) = \sum_n b_n e^{in\omega t}$$

and find that the equation of the forced oscillator becomes the equation

$$(in\omega)^2 a_n + \omega_0^2 a_n = \frac{b_n}{m}$$

or

$$a_n = \frac{b_n}{m (\omega_0^2 - n^2 \omega^2)}.$$  

In particular, if any multiple $n\omega$ of the frequency $\omega$ of the applied force $F(t)$ is very close to the frequency $\omega_0$ of the unforced oscillator, the term $b_n$ of that frequency in the force gives rise huge oscillations, called resonances. Note also that there is no freedom to pick these $a_n$ terms, i.e. the solution is unique, except if $\omega_0 = \omega$, in which case we can see that we must have all $b_n = 0$, i.e. no force, and then we are back with the old harmonic oscillator.

**7.12 Review**

Example:

$$f(x) = x \quad 0 \leq x < 2\pi$$

and $f(x)$ is $2\pi$ periodic. Find the complex amplitudes $c_k$ of $f(x)$ and determine the percentage of the energy of $f(x)$ stored in the amplitudes $c_{-1}, c_0, c_1$.

**Solution:** The integrals we need are

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} xe^{-ik\pi} dx$$

Integrating by parts:

$$\int xe^{\alpha x} dx = \frac{e^{\alpha x}}{\alpha} \left( x - \frac{1}{\alpha} \right) + c$$
if $\alpha \neq 0$, and if $\alpha = 0$,
\[
\int xe^{\alpha x} \, dx = \int xe^0 \, dx = \int x \, dx = \frac{x^2}{2} + c .
\]
Plugging this in gives
\[
c_k = \frac{1}{-ik} = \frac{i}{k}
\]
when $k \neq 0$, and
\[
c_0 = \pi .
\]
The energy in the function $f(x)$ is
\[
\int_0^{2\pi} f(x)^2 \, dx = \int_0^{2\pi} x^2 \, dx = \frac{8}{3}\pi^3
\]
The energy in the term $c_k$ is
\[
T |c_k|^2 = \frac{2\pi}{k^2}
\]
for $k \neq 0$, while
\[
T |c_0|^2 = 2\pi^3
\]
so that the fraction of the energy in the terms $c_{-1}, c_0, c_1$ is
\[
\frac{\frac{2\pi + 2\pi^3 + 2\pi}{\frac{8}{3}\pi^3}}{\frac{6 + 3\pi^2}{4\pi^2}} = \frac{6/\pi^2 + 3}{4} .
\]
We calculate
\[
\pi^2 = (3.141\ldots)^2 \sim 9.87 < 10
\]
so that the fraction of the energy in $c_{-1}, c_0, c_1$ is
\[
\% \text{ of energy in } c_{-1}, c_0, c_1 > \frac{6/\pi^2 + 3}{4} > \frac{6/10 + 3}{4} = \frac{9}{10} .
\]
See the picture in figure 22 \[\Box\]

You need to know how to compute some Fourier series, and to compute energy in the function and in the amplitudes.

Good review examples: textbook, page 25: example 1, page 29: example 2, page 32, exercises 5, 6, 7, 8, 9, 11, 13, 15, page 40: examples 4, 5, page 58: example 1 (you don’t need to know about the sinh function). In each of these, you should be able to work out the energy of the function, and what percentage of the energy is contained in the first few amplitudes (either the $a_m, b_m$ real amplitudes, or the $c_k$ complex amplitudes, depending on the problem).

IMPORTANT: you can ignore the textbook discussion of discontinuities. Just calculate the amplitudes for these examples, and make sure you get them right.

You can bring one sheet of paper with you to the test, with whatever you want written on one side of it.
Figure 22: We capture over 90% of the energy with just the $c_{-1}, c_0, c_1$ terms

Figure 23: A vibrating string, at initial time

8 Solving PDEs with Fourier series: waves and heat

Take a piece of string, as in figure 23 tied down on the x-axis at $x = 0$ and $x = L$. Suppose that

1. the string stays tied down at $x = 0$ and $x = L$
2. it vibrates only in the plane
3. its position at time $t$ is given as the graph of a function $u(x,t)$
4. the string density is a constant $\rho$
5. the string bends easily
6. vibrations are small, and the string stays nearly flat
7. string tension $\tau$ is nearly constant in magnitude (since the string length does not change much)

Pick a little piece of the string, as in figure 24. The tension vectors are drawn in figure 25. If $\theta(x)$ is the angle of the tangent to the string, (pointing forward along the string), then these tension vectors are

\[
\begin{align*}
(-\tau \cos \theta(x), -\tau \sin \theta(x)) & \text{ on the left end and} \\
(\tau \cos \theta(x + \Delta x), \tau \sin \theta(x + \Delta x)) & \text{ on the right end.}
\end{align*}
\]

So the total force on this bit of string is the sum of these. The vertical component of the force is

\[F = \tau \sin \theta(x + \Delta x) - \tau \sin \theta(x).\]

Since $F = ma$ we need $m$. We approximate $m \cong \rho \Delta x$ (mass is density times length), since the string is nearly flat, so length is approximately $\Delta x$ (as in figure 26 on the next page). Acceleration $a$ is second derivative of position with respect to time, so vertical acceleration is

\[a = \frac{\partial^2 u}{\partial t^2}.\]
Figure 26: Length of nearly flat piece of string is roughly $\Delta x$

Figure 27:

So finally we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{F}{m} = \frac{1}{\rho \Delta x} \left( \tau \sin \theta(x + \Delta x) - \tau \sin \theta(x) \right) = \frac{\tau}{\rho \Delta x} \left( \sin \theta(x + \Delta x) - \sin \theta(x) \right).$$

To approximate the $\sin \theta$ stuff, use the fact that the derivative is the tangent of the angle

$$\frac{\partial u}{\partial x} = \tan \theta(x).$$

Looking at figure 27, you see that

$$\sin \theta(x) = \frac{\tan \theta(x)}{\sqrt{1 + \tan^2 \theta(x)}} = \frac{\frac{\partial u}{\partial x}}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}}.$$
Expanding out the square root,
\[ \sin \theta(x) = \frac{\partial u}{\partial x} - \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^3 + \frac{3}{8} \left( \frac{\partial u}{\partial x} \right)^5 + \ldots. \]

Since the string is nearly flat, the slope is very small, i.e. \( \partial u / \partial x \) is very small, so we can drop the higher order terms, and get
\[ \sin \theta(x) = \frac{\partial u}{\partial x}. \]

Putting this into our equation for acceleration,
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \left( \frac{\partial u}{\partial x} (x + \Delta x) - \frac{\partial u}{\partial x} (x) \right) / \Delta x \]
and for \( \Delta x \) small this is approximately
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2}. \]

Let
\[ c = \sqrt{\frac{\tau}{\rho}}. \]

Call \( c \) the wave velocity. Then the wave equation is
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \]

**Solved Problems #4**

1. Derive the equation of vibration of a stretched string, taking only tension into account, with linear density \( 10^{-3} \text{ kg/m} \) and \( \tau = 100 \text{ N} \).

   **Solution:**
   \[ \frac{\partial^2 u}{\partial t^2} = 10^5 \frac{\partial^2 u}{\partial x^2} \]

   \( \square \)

2. The same again, but taking the weight into account.

   **Solution:**
   \[ \frac{\partial^2 u}{\partial t^2} = 10^5 \frac{\partial^2 u}{\partial x^2} - 9.8 \]

   \( \square \)

3. An elastic bar lying along the \( x \) axis has one end at the origin. Stretching the bar out a little along the \( x \) axis and letting it go, the bar vibrates horizontally, squishing and stretching. Suppose that the part of the bar that sits at a location \( x \) when the bar is in equilibrium (no forces applied)
sits instead at \( x + u(x, t) \) at time \( t \) when it is vibrating. Let \( F(x, t) \) be the force near this part of the bar, i.e. \( F(x, t) \) is the force exerted at time \( t \) on the part of the bar between position \( x + u(x, t) \) and \( x + u(x, t) + \Delta x \). Experiment shows that

\[
F(x, t) = -AE \frac{\partial u}{\partial x}
\]

where \( A \) is the area of a cross section of the bar (slicing it at any fixed \( x \)) and \( E \) is a positive constant called Young’s modulus. Let \( \rho \) be the mass density of the bar.

(a) From \( F = ma \) derive

\[
\rho A \Delta x \frac{\partial^2 u}{\partial t^2} = -AE \left. \frac{\partial u}{\partial x} \right|_x + AE \left. \frac{\partial u}{\partial x} \right|_{x+\Delta x}.
\]

(b) Show that

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

where

\[
c^2 = \frac{E}{\rho}.
\]

Solution:

(a) We apply \( F = ma \). The force exerted on the portion of bar which (when at equilibrium) would be lying between \( x \) and \( x + \Delta x \) is

\[
- AE \frac{\partial u}{\partial x}(x, t)
\]

on the left side (pulling to the left) and

\[
AE \frac{\partial u}{\partial x}(x + \Delta x, t)
\]

on the right side (pulling right). The resulting force is thus

\[
F = - AE \frac{\partial u}{\partial x}(x, t) + AE \frac{\partial u}{\partial x}(x + \Delta x, t).
\]

The mass is

\[
m = \rho A \cdot \text{length}
\]

but since the bar is only slightly stretched, we can say that its length is approximately

\[
\text{length} \sim \Delta x
\]

so that

\[
m = \rho A \Delta x.
\]
The acceleration is \( \frac{\partial^2 u}{\partial t^2} \)

so putting this together

\[
\rho A \Delta x \frac{\partial^2 u}{\partial t^2} = -AE \frac{\partial u}{\partial x} (x, t) + AE \frac{\partial u}{\partial x} (x + \Delta x, t).
\]

(b) Divide by \( \rho A \Delta x \):

\[
\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \left( \frac{\partial u}{\partial x} (x + \Delta x, t) - \frac{\partial u}{\partial x} (x, t) \right) \Delta x.
\]

Take \( \Delta x \to 0 \):

\[
\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2}
\]

or, if we define a constant \( c \) by

\[
c = \sqrt{\frac{E}{\rho}}
\]

then

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.
\]

\[\square\]

4. Take a rectangular sheet and set it vibrating. Suppose that the vibrations are pretty much entirely vertical, that the tension is constant. Take coordinates \( x, y, z \) so that two sides of the sheet lie along the \( x \) and \( y \) axes. Suppose that one corner lies at the origin, and the side lengths are \( \Delta x \) and \( \Delta y \). Let \( \rho \) be the mass density of the sheet (a constant).

(a) Show that the vertical component of the tension vector coming from the opposite edges of the sheet are roughly

\[
\tau \Delta x \left( \frac{\partial u}{\partial y} \bigg|_{y=L_Y} - \frac{\partial u}{\partial y} \bigg|_{y=0} \right)
\]

and

\[
\tau \Delta y \left( \frac{\partial u}{\partial x} \bigg|_{x=L_X} - \frac{\partial u}{\partial x} \bigg|_{x=0} \right).
\]

(b) If no force acts on the sheet other than tension, show that

\[
\frac{\partial^2 u}{\partial t^2} \rho \Delta x \Delta y = \tau \Delta x \left( \frac{\partial u}{\partial y} \bigg|_{y=L_Y} - \frac{\partial u}{\partial y} \bigg|_{y=0} \right) + \tau \Delta y \left( \frac{\partial u}{\partial x} \bigg|_{x=L_X} - \frac{\partial u}{\partial x} \bigg|_{x=0} \right).
\]
Derive from this
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]
(the wave equation) where
\[ c^2 = \frac{\tau}{\rho}. \]

**Solution:**

5. The picture is not very good. Perhaps figure 28 on the next page is clearer.
Figure 28: The vibrating sheet. Here $\gamma$ is the angle of the tangent vector perpendicular to the sheet. The corner $A$ is at $(x, y)$ while the corner $B$ is at $(x + \Delta x, y + \Delta y)$. 
Let us just look at the edge $AB$. The force vector at a point of that edge has length $\tau \Delta x$ and has some angle $\gamma$ from the horizontal, and is perpendicular to the edge $AB$ itself. Since this little piece of sheet is assumed nearly flat (because the vibrations are small), it will have tangent vector close to $(0, -1, 0)$.

The actual tangent vector some point of the edge $AB$ will be perpendicular to the $x$ direction (because that is the direction of the edge $AB$). So this force vector lives in the $y, z$ directions only (where $z$ is the vertical direction). Therefore the force vector must be

$$F_{AB} = \tau \Delta x (0, \cos \gamma, \sin \gamma).$$

This $\gamma$ is just the angle of the tangent line along the $-y$ direction:

$$\tan \gamma = -\frac{\partial u}{\partial y}.$$

With a little trigonometry (using the fact that $\gamma \sim \pi$)

$$\cos \gamma = \frac{-1}{\sqrt{1 + \left(\frac{\partial u}{\partial y}\right)^2}} \quad \sin \gamma = -\frac{\frac{\partial u}{\partial y}}{\sqrt{1 + \left(\frac{\partial u}{\partial y}\right)^2}}$$

so that up to second order in $\partial u/\partial x$ (which is small, because the vibrations are small) the force is

$$F_{AB} = -\tau \Delta x \left(0, 1, \frac{\partial u}{\partial y}(x, y)\right).$$

For all of the other edges, similar analysis applies; but for example on $CD$ the force pulls in the opposite direction, in the $y$ direction rather than the $-y$ direction. Hence

$$F_{CD} = \tau \Delta x \left(0, 1, \frac{\partial u}{\partial y}(x, y + \Delta y)\right).$$

Therefore the resultant force on the front and back sides is

$$F_{AB} + F_{CD} = \tau \Delta x \left(0, 0, \frac{\partial u}{\partial y}(x, y + \Delta y) - \frac{\partial u}{\partial y}(x, y)\right)$$

giving vertical force

$$\tau \Delta x \left(\frac{\partial u}{\partial y}(x, y + \Delta y) - \frac{\partial u}{\partial y}(x, y)\right).$$

Flipping $x$ and $y$ coordinates (just notation) we see that

$$F_{AD} + F_{BC} = \tau \Delta y \left(0, 0, \frac{\partial u}{\partial x}(x + \Delta x, y) - \frac{\partial u}{\partial x}(x, y)\right).$$
Taking the vertical part we get
\[ \tau \Delta y \left( \frac{\partial u}{\partial x} (x + \Delta x, y) - \frac{\partial u}{\partial x} (x, y) \right) \]

6. \( F = ma \) and from part (a) we have
\[ F = \tau \left( \frac{\partial u}{\partial y} (x, y + \Delta y) - \frac{\partial u}{\partial y} (x, y) \right) \Delta x + \tau \left( \frac{\partial u}{\partial x} (x + \Delta x, y) - \frac{\partial u}{\partial x} (x, y) \right) \Delta y. \]

The mass is
\[ m = \rho \cdot \text{Area} \]
which, since the sheet is nearly flat (the vibrations are small), is approximately
\[ m = \rho \Delta x \Delta y. \]
The acceleration is
\[ \frac{\partial^2 u}{\partial t^2} \]
so putting this together:
\[ \Delta x \Delta y \rho \frac{\partial^2 u}{\partial t^2} = \tau \left( \frac{\partial u}{\partial y} (x, y + \Delta y) - \frac{\partial u}{\partial y} (x, y) \right) \Delta x + \tau \left( \frac{\partial u}{\partial x} (x + \Delta x, y) - \frac{\partial u}{\partial x} (x, y) \right) \Delta y. \]
Divide by \( \Delta x \Delta y \rho \) to find
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial u}{\partial y} (x, y + \Delta y) - \frac{\partial u}{\partial y} (x, y) \Delta y + \frac{\tau}{\rho} \frac{\partial u}{\partial x} (x + \Delta x, y) - \frac{\partial u}{\partial x} (x, y) \Delta x. \]
Let \( \Delta x \to 0 \) and \( \Delta y \to 0 \). You get
\[ \frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \]
Let
\[ c = \sqrt{\frac{\tau}{\rho}}. \]
3.
Solution:
\[ L = 1 \]
\[ c = 1 \]
\[ \lambda_n = \frac{c \pi n}{L} \]
\[ = \pi n \]
\[ b_n = 0 \quad (\text{because } g(x) = 0) \]
\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{\pi nx}{L} \right) dx \]
These \( b_n \) are simply the Fourier amplitudes of \( f(x) \), which is

\[
f(x) = \sin(\pi x) + 3\sin(2\pi x) - \sin(5\pi x)
\]

so its amplitudes are obviously

\[
\begin{align*}
  b_1 &= 1 \\
  b_2 &= 3 \\
  b_5 &= -1
\end{align*}
\]

and all other \( b_n \) are 0. Therefore

\[
u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{\pi nx}{L}\right) b_n \cos(\lambda_n t)
\]

\[
= \sin(\pi x) \cos(\pi t) + 3\sin(2\pi x) \cos(2\pi t) - \sin(5\pi x) \cos(5\pi t)
\]

\( \square \)

9.

Solution:

\[
L = 1 \\
c = 1 \\
f(x) = x(1 - x) \\
g(x) = \sin(\pi x) \\
\lambda_n = \frac{\pi nc}{L} \\
= \pi n
\]

\[
b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi nx}{L}\right) \, dx
\]

\[
= 2 \int_0^1 x(1 - x) \sin(\pi nx) \, dx
\]

Carrying out integration by parts, one eventually gets

\[
b_n = 4 \frac{1 - (-1)^n}{\pi^3 n^3}.
\]

while for the \( b_n^* \) we have

\[
b_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{\pi nx}{L}\right) \, dx
\]

\[
= \frac{2}{n\pi} \int_0^1 \sin(\pi x) \sin(\pi nx) \, dx
\]

\[
= \begin{cases}
  \frac{1}{2} & n = 1 \\
  0 & n \neq 1
\end{cases}
\]

66
so that finally

\[ b_1^* = \frac{1}{\pi} \]

while all other \( b_n^* \) are 0. This gives

\[ u(x, t) = \frac{1}{\pi} \sin(\pi x) \sin(\pi t) + \sum_{n=1}^{\infty} \sin(\pi n x) \frac{4}{\pi^3} \frac{(-1)^n}{n^3} \cos(\pi n t) \]

or you could plug in \( n = 2k + 1 \) and sum over all \( k = 0, 1, \ldots \), since even values of \( n \) will actually not contribute to this sum; then you would get the answer from the textbook. (But the above result is good enough.)

\[ \square \]

8.1 Fourier and d’Alembert pictures of waves

Lecture 9

Two pictures of waves: see figure 29 on the following page. A vibrating string can do both of these things. But it can do other things too. Idea: every wave motion can be broken up (1) (as in Fourier series) into a sum of infinitely many waves behaving like the Fourier series picture OR (2) into a sum of 2 waves, one travelling to the left, the other to the right, as in the d’Alembert picture.

8.1.1 The Fourier picture

In Fourier’s picture, a wave just bounces up and down, keeping the same shape. Suppose the shape is given by the graph of a function \( F(x) \). Over time, this graph gets rescaled, by a factor \( G(t) \); so it keeps the same shape, just rescaled. Thus the height of the wave is

\[ u(x, t) = F(x)G(t). \]

We expect \( G(t) \) to bounce up and down. But also \( u(x, t) \) satisfies the wave equation

\[ 0 = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F \frac{d^2 G}{dt^2} - c^2 \frac{d^2 F}{dx^2} G. \]

Divide by \( FG \) to get

\[ \frac{1}{G} \frac{d^2 G}{dt^2} = \frac{c^2}{F} \frac{d^2 F}{dx^2} \]

only depends on \( t \)

only depends on \( x \)
(a) Fourier’s picture of a wave: bouncing up and down

(b) d’Alembert’s picture of a wave: travelling waves

Figure 29: Two ways of thinking about waves
But the left side depends only on $t$, while the right side depends only on $x$, and they are equal. So both must be constant, say $-k$ for some number $k$. This gives

$$ \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{k}{c^2} $$

$$ \frac{1}{G} \frac{d^2 G}{dt^2} = -k. $$

Rewrite this as

$$ \frac{d^2 F}{dx^2} = -\frac{k}{c^2} F $$

$$ \frac{d^2 G}{dt^2} = -kG $$

harmonic oscillators. Since $F$ (the shape of the string) vanishes at $x = 0$ and $x = L$ (where it is tied down), $F$ has two zeros. This forces the spring constant $k/c^2$ to be positive, so $k > 0$, and then the general solution of the harmonic oscillator equation for $F$ is

$$ F(x) = a \cos \left( \sqrt{\frac{k}{c^2}} x \right) + b \sin \left( \sqrt{\frac{k}{c^2}} x \right). $$

To get $F = 0$ at $x = 0$ we need $a = 0$, so just

$$ F(x) = b \sin \left( \sqrt{\frac{k}{c^2}} x \right). $$

We can rescale the shape $F(x)$ to get $b = 1$. To get $F = 0$ at $x = L$, we need

$$ 0 = \sin \left( \sqrt{\frac{k}{c^2}} L \right) $$

so that

$$ \sqrt{\frac{k}{c^2}} L = \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \ldots. $$

Lets say

$$ \sqrt{\frac{k}{c^2}} L = \pi m $$

where $m$ is a positive integer (it has to be positive since the left hand side is positive). Solve for $k$:

$$ k = \frac{\pi^2 m^2 c^2}{L^2}. $$

The frequency of this wave $F(x)$ is then

$$ p = \frac{\pi m}{L}. $$
Now plug into the harmonic oscillator equation for $G(t)$:

$$\frac{d^2G}{dt^2} = -kG = -\frac{\pi^2mc^2}{L^2}.$$ 

So $G$ is a harmonic oscillator of frequency

$$\omega = \frac{\pi mc}{L}.$$ 

So the oscillation in time is

$$G(t) = A \cos\left(\frac{\pi mc}{L} t\right) + B \sin\left(\frac{\pi mc}{L} t\right).$$ 

Putting this together, the final wave is $u(x, t) = FG = or explicitly$

$$u(x, t) = \sin\left(\frac{\pi m}{L} x\right) \left(A \cos\left(\frac{\pi mc}{L} t\right) + B \sin\left(\frac{\pi mc}{L} t\right)\right).$$ 

This is exactly the Fourier picture: a sinusoidal oscillation of a spatial sine wave. Looks like a plucked guitar string, playing a particular note. To get a higher note, i.e. faster temporal frequency, just use a shorter string (a fret on the guitar), making $L$ smaller, and the frequency of vibration higher. 

### 8.2 Making any wave out of Fourier’s pure notes

Add together pure notes of the form found by Fourier, at different frequencies:

$$u(x, t) = \sum_{m=1}^{\infty} \sin\left(\frac{\pi m}{L} x\right) \left(A_m \cos\left(\frac{\pi mc}{L} t\right) + B_m \sin\left(\frac{\pi mc}{L} t\right)\right).$$ 

This is the general solution of the wave equation: all solutions look like this for some choice of amplitudes $A_m$ and $B_m$. 

### 8.2.1 Digging the amplitudes out of the physical data

Plug in $t = 0$ to our general solution, we get

$$u|_{t=0} = \sum_{m=1}^{\infty} A_m \sin\left(\frac{\pi m}{L} x\right)$$

because $\sin(0) = 0$ and $\cos(0) = 1$. This is just a Fourier series, in $\sin$ so we use our usual formulas:

$$A_m = \frac{2}{L} \int_{0}^{L} u|_{t=0} \sin\left(\frac{\pi nx}{L}\right) dx.$$ 

How do we find the $B'$s? Take the velocity of the wave, $\partial u/\partial t$,

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} \sin\left(\frac{\pi m}{L} x\right) \left(-\frac{\pi mc}{L} A_m \sin\left(\frac{\pi mc}{L} t\right) + \frac{\pi mc}{L} B_m \cos\left(\frac{\pi mc}{L} t\right)\right)$$
(because $\sin' = \cos$, etc.) Now look at time $t = 0$:

$$\frac{\partial u}{\partial t} \bigg|_{t=0} = \sum_{m=1}^{\infty} \sin \left( \frac{\pi m}{L} x \right) \left( \frac{\pi mc}{L} \right) B_m.$$ 

As a Fourier series, we can recover the amplitudes from the usual formula:

$$\left( \frac{\pi mc}{L} \right) B_m = \frac{2}{L} \int_0^L \frac{\partial u}{\partial t} \bigg|_{t=0} \sin \left( \frac{\pi nx}{L} \right) \, dx$$

which we solve for $B_m$ to get

$$B_m = \frac{2}{\pi mc} \int_0^L \frac{\partial u}{\partial t} \bigg|_{t=0} \sin \left( \frac{\pi nx}{L} \right) \, dx.$$

### 8.3 d’Alembert’s picture

Pick a shape for a wave to have, say $f(x)$. Then to make the shape move $x_0$ units to the right, plug in $x - x_0$, i.e. write down $f(x - x_0)$. When you plug $x = x_0$ into the function $f(x - x_0)$ you get $f(0)$, so the value of $f(x)$ from the origin has moved to $x = x_0$. Therefore we have shifted the graph forward a distance of $x_0$.

To make it move forward continuously, with velocity $c$, we plug in $x_0 = ct$. So look at the function $f(x - ct)$. At time $t = 0$, this is just old $f(x)$. But at later times, it is the same graph moved to the right $ct$ units.

Now check with the chain rule that if $u(x,t) = f(x - ct)$ then $u$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Why?

- $u = f(x - ct)$
- $\frac{\partial u}{\partial t} = f'(x - ct) \frac{d}{dt}(x - ct) = f'(x - ct)(-c)$
- $\frac{\partial^2 u}{\partial t^2} = f''(x - ct)(-c)^2 = c^2 f''(x - ct)$
- $\frac{\partial u}{\partial x} = f'(x - ct)$
- $\frac{\partial^2 u}{\partial x^2} = f''(x - ct)$.

So it is clear that $u$ satisfies the wave equation.

Similarly, $u(x,t) = f(x + ct)$ is a left moving wave, and satisfies the wave equation.
8.4 Connecting the two pictures

Suppose we use Fourier’s technique to write

\[ u(x, t) = \sum_{m=1}^{\infty} \sin \left( \frac{\pi m}{L} x \right) \left( A_m \cos \left( \frac{\pi mc}{L} t \right) + B_m \sin \left( \frac{\pi mc}{L} t \right) \right). \]

Trigonometric identities:

\[
\begin{align*}
\sin(A + B) &= \sin(A) \cos(B) + \cos(A) \sin(B) \\
\cos(A + B) &= \cos(A) \cos(B) - \sin(A) \sin(B) \\
\sin(A - B) &= \sin(A) \cos(B) - \cos(A) \sin(B) \\
\cos(A - B) &= \cos(A) \cos(B) - \sin(A) \sin(B).
\end{align*}
\]

We can use these to show that

\[
\begin{align*}
\sin(A) \cos(B) &= \frac{1}{2} (\sin(A + B) + \sin(A - B)) \\
\sin(A) \sin(B) &= \frac{1}{2} (\cos(A - B) - \cos(A + B)).
\end{align*}
\]

Now use this in the Fourier series to find

\[
\sin \left( \frac{\pi mx}{L} \right) \cos \left( \frac{\pi mct}{L} \right) = \frac{1}{2} \sin \left( \frac{\pi m}{L} (x + ct) \right) + \frac{1}{2} \sin \left( \frac{\pi m}{L} (x - ct) \right).
\]

Similarly for the other terms.

Rewriting the whole expansion in terms of left and right movers,

\[
u(x, t) = \frac{1}{2} \sum_{m=1}^{\infty} \left( A_m \sin \left( \frac{\pi m}{L} (x + ct) \right) - B_m \cos \left( \frac{\pi m}{L} (x + ct) \right) \right) + \frac{1}{2} \sum_{m=1}^{\infty} \left( A_m \sin \left( \frac{\pi m}{L} (x - ct) \right) + B_m \cos \left( \frac{\pi m}{L} (x - ct) \right) \right).
\]

Claim: I can write this simply as

\[
u(x, t) = \frac{1}{2} u(x + ct, 0) + \frac{1}{2} u(x - ct, 0) + \frac{1}{2c} \int_{x-ct}^{x+ct} u \left( x, t \right) \frac{\partial u}{\partial t} \bigg|_{t=0} \, dx.
\]

8.5 Heat

Take a piece of wire of length \( L \), and lay it along the \( x \) axis from \( x = 0 \) to \( x = L \). Let \( u(x, t) \) be the temperature of the wire at point \( x \) at time \( t \). Suppose the wire is in an insulated tube, so heat can only travel along the wire. Fourier’s
law: the cold parts heat and the hot parts cool. So heat moves toward the cold. For example in figure 30 the heat increases to the right, so the heat must flow to the left: heat flux is negative. But the temperature increasing to the right means exactly
\[ \frac{\partial u}{\partial x} > 0. \]
So heat flux is negative where \( \frac{\partial u}{\partial x} \) is positive, and vice versa. The simplest way to get this to work is to guess that
\[ \text{heat flux} = -K \frac{\partial u}{\partial x}, \]
where \( K \) is some positive constant.

Now a little piece of wire, between \( x \) and \( x + \Delta x \), gains heat from flow across \( x \) and looses it from flow out of \( x + \Delta x \), so if \( Q \) is total heat, then total heat changes in a small amount \( \Delta t \) of time by
\[ \Delta Q = \Delta t K \left( \frac{\partial u}{\partial x} \bigg|_{x+\Delta x} - \frac{\partial u}{\partial x} \bigg|_x \right). \]
But total heat in this piece is just
\[ Q = a \int_{x}^{x+\Delta x} u \, dx \]
where \( a \) is a constant called “heat capacity.” So heat changes by
\[ \Delta Q = Q \bigg|_{t+\Delta t} - Q \bigg|_t = a \int_{x}^{x+\Delta x} u(x, t + \Delta t) \, dx - a \int_{x}^{x+\Delta x} u(x, t) \, dx. \]
(I am using \( x \) inside the integral as variable of integration, which could get confused with the \( x \) representing the left end of the piece of wire. Be careful.) Putting these together
\[ \Delta t K \left( \frac{\partial u}{\partial x} (x + \Delta x, t) - \frac{\partial u}{\partial x} (x, t) \right) \, dt = a \int_{x}^{x+\Delta x} (u(x, t + \Delta t) - u(x, t)) \, dx. \]
Now divide by $\Delta t \Delta x$ to find
\[
K \frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) = \frac{1}{\Delta x} \int_x^{x+\Delta x} a \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} \, dx.
\]
But the expression
\[
\frac{1}{\Delta x} \int_x^{x+\Delta x}
\]
gives the average value between $x$ and $x + \Delta x$. So as we make $\Delta x$ small, this is just the value at $x$. If we make both $\Delta t$ and $\Delta x$ small, the equation becomes
\[
K \frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t}.
\]
Define $c = \sqrt{K/a}$ and then we have the heat equation
\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.
\]

### 8.5.1 Separating variables

As for the wave equation, we look first form solutions $u(x, t)$ of the wave equation which have a fixed shape $F(x)$ which gets rescaled over time, $u(x, t) = F(x)G(t)$. We will suppose that the ends of the wire are fixed at $0^\circ C$ using a thermostat. So $F(0) = F(L) = 0$. Plugging in $u(x, t) = F(x)G(t)$ to the heat equation gives
\[
\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}.
\]
We have only $t$ on the left, and only $x$ on the right, so both sides must be constant, say $-k$. We find that $F(x)$ is a harmonic oscillator, and so as for the wave equation
\[
F(x) = \sin \left( \frac{\pi N x}{L} \right)
\]
and the spring constant $k$ is
\[
k = \left( \frac{\pi N}{L} \right)^2.
\]
This $N$ can be any positive integer. (Because $\sin(-x) = -\sin(x)$, using negative integers just gives functions $F(x)$ with the same shapes, upside down, and we can just absorb that minus sign into $G(t)$; using $N = 0$ gives $F(x) = 0$, nothing.)

As for $G(t)$ we find the equation
\[
G'(t) = -c^2 k G(t) = -c^2 \left( \frac{\pi N}{L} \right)^2 G(t).
\]
This is just an exponential decay, so the solution is

\[ G(t) = A \exp\left(-\left(\frac{\pi N c}{L}\right)^2 t\right) \]

where the amplitude \( A \) can be any real number. So the resulting mode of the heat equation is

\[ u(x, t) = F(x)G(t) = A \exp\left(-\left(\frac{\pi N c}{L}\right)^2 t\right) \sin\left(\frac{\pi N x}{L}\right) \]

where \( A \) is any real number and \( N \) any positive integer. It looks like figure 31.

### 8.5.2 The general solution of the heat equation

The general solution is a sum of modes of all frequencies

\[ u(x, t) = \sum_{n=1}^{\infty} A_n \exp\left(-\left(\frac{\pi N c}{L}\right)^2 t\right) \sin\left(\frac{\pi n x}{L}\right). \]

To find the amplitudes \( A_n \), take \( t = 0 \) and you get

\[ u(x, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{L}\right), \]

just a Fourier sine series. So the amplitudes are given by the usual formula:

\[ A_n = \frac{2}{L} \int_0^L u(x, t = 0) \sin\left(\frac{\pi n x}{L}\right) \, dx. \]
Conceptually, each mode is a sine wave in space, decaying exponentially at a rate
\[
\left( \frac{\pi n c}{L} \right)^2
\]
so high frequencies (large \( n \)) decay extremely quickly, almost at once. Soon only the low frequencies are left:
\[
u(x, t) = A_1 \exp \left( - \left( \frac{\pi c}{L} \right)^2 t \right) \sin \left( \frac{\pi x}{L} \right) + \text{small stuff}
\]
for large time \( t \), and even this term is decaying away exponentially.

### 8.5.3 Different thermostat temperatures at the ends

Lecture 12

Last time, our wire was at 0° at both ends. Suppose the wire has temperatures \( T_1 \) at \( x = 0 \) and \( T_2 \) at \( x = L \). Trick: suppose that \( U(x, t) \) is the temperature of the wire, and write
\[
U(x, t) = T_1 + \frac{T_2 - T_1}{L} x + u(x, t).
\]

The first two terms are just the linear function that hits \( T_1 \) at \( x = 0 \) and \( T_2 \) at \( x = L \). Check that this \( u(x, t) \) solves the heat equation, with \( u(x, t) = 0 \) at \( x = 0 \) and \( x = L \). The function \( u(x, t) \) is just \( U(x, t) \) tilted by adding a linear function to get the ends to go to zero. Now we can just use for \( u(x, t) \) the general solution from last time.

How did I come up with this trick? The idea is that this linear function satisfies the heat equation, and doesn’t change in time: it is a steady state of the heat equation. The function \( U(x, t) \) is expressed as a steady state of heat, plus a small fluctuation \( u(x, t) \), which we write in Fourier series. This is a big idea: solutions are steady states plus fluctuations.

What are all of the steady states? To have a steady state, we need to solve the heat equation
\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}
\]
with steady state, i.e. not changing in time, so
\[
\frac{\partial u}{\partial t} = 0.
\]
Putting this together, we need \( u(x, t) \) to be independent of \( t \) and to satisfy
\[
\frac{\partial^2 u}{\partial x^2} = 0.
\]
No acceleration implies a linear function, so we see that the steady states are exactly the linear functions.
8.5.4 Insulated ends

Suppose that instead of a thermostat at each end, we have insulation. So there is no heat flux at the ends:

\[ 0 = \text{heat flux} = -K \frac{\partial u}{\partial x} \]

at \( x = 0 \) and \( x = L \). Once again we separate variables, and get the same equations as before

\[ \frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \]

which leads to the same conclusion: both sides are constant, say \( k \), and \( F(x) \) is a harmonic oscillator, and \( G(t) \) is an exponential growth or decay. But is the spring constant \( k \) positive, negative, or zero? Because \( F'(x) = 0 \) at both ends (no heat flux), checking the possibilities

\[
F(x) = \begin{cases} 
A \cos \sqrt{k}x + B \sin \sqrt{k}x & \text{if } k > 0 \\
A + Bx & \text{if } k = 0 \\
A \cosh \sqrt{|k|}x + B \sinh \sqrt{|k|}x & \text{if } k < 0 
\end{cases}
\]

we find that the derivatives are

\[
F'(x) = \begin{cases} 
-\sqrt{k}A \sin \sqrt{k}x + \sqrt{k}B \cos \sqrt{k}x & \text{if } k > 0 \\
B & \text{if } k = 0 \\
\sqrt{|k|}A \sinh \sqrt{|k|}x + \sqrt{|k|}B \cosh \sqrt{|k|}x & \text{if } k < 0 
\end{cases}
\]

Plugging in the fact that \( F'(0) = 0 \), we get

\[
0 = \begin{cases} 
\sqrt{k}B & \text{if } k > 0 \\
B & \text{if } k = 0 \\
\sqrt{|k|}B \cosh \sqrt{|k|}x & \text{if } k < 0 
\end{cases}
\]

But since \( \cosh \) has no zeros, we find that \( k \geq 0 \) and \( B = 0 \), and so, after rescaling \( F(x) \) we get

\[
F(x) = \begin{cases} 
\cos \sqrt{k}x & \text{if } k > 0 \\
1 & \text{if } k = 0 
\end{cases}
\]

Now to get the \( F'(x) = 0 \) at \( x = L \), we see as we did for the wave equation that

\[
F(x) = \cos \left( \frac{\pi Nx}{L} \right)
\]

where \( N \) can be any integer. We can now allow \( N = 0 \) to get the spring constant to be \( k = 0 \). Solving the exponential decay equation for \( G(t) \) exactly as in the thermostat case,

\[
G(t) = A \exp \left( - \left( \frac{\pi Nc}{L} \right)^2 t \right)
\]
so that our mode is

\[ u(x, t) = A \exp \left( -\left( \frac{\pi N c}{L} \right)^2 t \right) \cos \left( \frac{\pi N x}{L} \right). \]

The general solution is as before a sum

\[ u(x, t) = \sum_{n=0}^{\infty} A_n \exp \left( -\left( \frac{\pi n c}{L} \right)^2 t \right) \cos \left( \frac{\pi n x}{L} \right) \]

and at \( t = 0 \) it is a Fourier cosine series

\[ u(x, 0) = \sum_{n=0}^{\infty} A_n \cos \left( \frac{\pi n x}{L} \right) \]

so that

\[ A_0 = \frac{1}{L} \int_0^L u(x, 0) \, dx \]

and

\[ A_n = \frac{2}{L} \int_0^L u(x, 0) \cos \left( \frac{\pi n x}{L} \right) \, dx. \]

Again, all high frequencies decay quickly, except the \( n = 0 \) frequency, which is just a constant \( A_0 \), the average temperature. So physically, the temperature rapidly reaches the equilibrium temperature, which is just the average temperature. See figure 32.

**Solved Problems #5**

1. Find the temperature \( u(x, t) \) of a wire with insulated ends, length \( L = \pi \), conductivity \( c = 1 \) and initial temperature \( f(x) = 100 \).
Solution: The wire stays at 100\(^0\) everywire along the wire for all time. □

2. Find the temperature of a wire with ends kept at 0\(^0\) with length \(L = \pi\), with \(c = 1\) and initial temperature \(f(x) = x\) inside the wire. What happens to the temperature after a long time?

Solution:

\[
L = \pi
\]
\[
c = 1
\]
\[
\lambda_n = \frac{c\pi n}{L} = n
\]
\[
b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi nx}{L}\right) \, dx
\]
\[
= \frac{2}{\pi} \int_0^\pi x \sin (nx) \, dx
\]
\[
= 2 \frac{(-1)^{n+1}}{n}
\]
\[
u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin\left(\frac{\pi nx}{L}\right)
\]
\[
= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 t} \sin(nx)
\]

As the modes decay away, eventually the temperature goes to 0\(^0\). □

3. Find the temperature of the same wire as the last problem, but with insulated ends. What happens to the temperature after a long time?
Solution:

\[ a_0 = \frac{1}{L} \int_0^L f(x) \, dx = \frac{1}{\pi} \int_0^\pi x \, dx = \frac{1}{\pi} \frac{\pi^2}{2} = \frac{\pi}{2} \]

\[ a_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{\pi nx}{L} \right) \, dx = \frac{2}{\pi} \int_0^\pi x \cos (nx) \, dx = \frac{2 (-1)^n - 1}{\pi n^2} \]

\[ u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos \left( \frac{\pi nx}{L} \right) \]

\[ = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} e^{-n^2 t} \cos (nx) \]

As the modes decay away, the temperature approaches the average temperature of

\[ f(x) \rightarrow a_0 = \frac{\pi}{2} \]

\[ \square \]

4. The total heat in a wire at time \( t \) is

\[ \int_0^L u(x, t) \, dx . \]

Show that with insulated ends, the total heat is conserved. (Hint: you differentiate the total heat in time, bringing the derivative under the integral sign, and then use the heat equation to turn \( t \) derivatives into \( x \) derivatives. Then you use the fundamental theorem of calculus: the integral of the derivative . . . .)

Solution: The total heat at time \( t \) is

\[ Q(t) = \int_0^L u(x, t) \, dx \]
so that its rate of change is

\[
\frac{dQ}{dt} = \frac{d}{dt} \int_0^L u(x, t) \, dx
= \int_0^L \frac{\partial u}{\partial t} \, dx
= \int_0^L c^2 \frac{\partial^2 u}{\partial x^2} \, dx
= c^2 \int_0^L \frac{\partial}{\partial x} \frac{\partial u}{\partial x} \, dx
= c^2 \frac{\partial u}{\partial x} \bigg|_{x=0}^{x=L}
\]

Since there is no heat flux through the boundary (because it is insulated) and since by Fourier’s law, heat flux is

\[
\text{Heat flux} = -K \frac{\partial u}{\partial x}
\]

for \( K \) some positive constant, we must have

\[
\frac{\partial u}{\partial x} = 0
\]
at the boundary (i.e. at \( x = 0 \) and \( x = L \)). Therefore,

\[
\frac{dQ}{dt} = 0
\]

and so \( Q(t) \) is a constant. \( \square \)

5. What happens to the heat equation if we allow heat to escape sideways through the insulating tube around the wire?

**Solution:**

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} - \lambda (u - T_0)
\]

where \( \lambda \) is a positive constant measuring how poor the insulation is, and \( T_0 \) is the ambient temperature. \( \square \)

8.6 Heat and waves in square plates

8.6.1 The vibrating sheet

Similar to the vibrating string, the vibrating sheet satisfies

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).
\]
Let’s tie down the sheet at the edges of a rectangle, as in Figure 33. We will take our rectangle to be $0 \leq x \leq a, 0 \leq y \leq b$. We measure the initial position $u(x, y, t = 0)$ and initial velocity $\frac{\partial u}{\partial t}(x, y, t = 0)$. We want to know what happens at later times. As before, we start by looking for solutions which are products of functions of one variable each.

$$u(x, t) = E(x)F(y)G(t)$$

of functions of one variable each. Plugging into the wave equation, you get

$$\frac{G''(t)}{c^2G(t)} = \frac{E''(x)}{E(x)} + \frac{F''(y)}{F(y)}.$$ 

The left hand side has no $x$ or $y$ in it, and the right hand side has no $t$ in it, so both sides are constant, say equal to $-\alpha$.

$$\frac{G''(t)}{c^2G(t)} = -\alpha = \frac{E''(x)}{E(x)} + \frac{F''(y)}{F(y)}.$$ 

The left hand side is just an oscillator equation, but what about the right hand side? It is

$$-\alpha = \frac{E''(x)}{E(x)} + \frac{F''(y)}{F(y)}$$

and can be written

$$\frac{E''(x)}{E(x)} = -\alpha - \frac{F''(y)}{F(y)}.$$ 

Now the left hand side has no $y$ in it, and the right hand side has no $x$, so both are constant, say equal to $-\beta$. Then we get three harmonic oscillators

$$0 = G''(t) + c^2\alpha G(t)$$
$$0 = E''(x) + \beta E(x)$$
$$0 = F''(y) + (\alpha - \beta)F(y)$$
To get the sheet to stay tied down at $x = 0$ and $x = a$ we need $E(0) = E(a) = 0$; similarly we need $F(0) = F(b) = 0$. So we must have $\beta > 0$ and $\alpha - \beta > 0$. As before, this gets us

$$E(x) = \sin \left( \frac{\pi mx}{a} \right)$$

$$F(y) = \sin \left( \frac{\pi ny}{b} \right)$$

for some integers $m$ and $n$, and the spring constants must be

$$\beta = \left( \frac{\pi m}{a} \right)^2$$

$$\alpha - \beta = \left( \frac{\pi m}{a} \right)^2.$$

Finally, we find the spring constant for $G$ is $c^2 \alpha$, and $G(t)$ doesn’t have to vanish at any end points, so

$$G(t) = A \cos (\omega t) + B \sin (\omega t)$$

where $\omega$, the frequency, is

$$\omega = \sqrt{c^2 \alpha}$$

(the square root of the spring constant). Working this out,

$$\omega = c \pi \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2}.$$

So a single mode looks like

$$u(x, t) = E(x)F(y)G(t)$$

$$= \sin \left( \frac{\pi mx}{a} \right) \sin \left( \frac{\pi ny}{b} \right) (A \cos (\omega t) + B \sin (\omega t))$$

with

$$\omega = c \pi \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2}.$$

As usual, the general solution is just a sum of modes:

$$u(x, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \left( \frac{\pi mx}{a} \right) \sin \left( \frac{\pi ny}{b} \right) (A_{mn} \cos (\omega_{mn} t) + B_{mn} \sin (\omega_{mn} t))$$

where

$$\omega_{mn} = c \pi \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2}.$$
8.6.2 How to compute the amplitudes

Plugging in $t = 0$ gives a double Fourier series

$$u(x, y, 0) = \sum_{m, n=1}^{\infty} A_{mn} \sin \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right).$$

Using usual Fourier series techniques in first $x$ and then in $y$ we get

$$A_{mn} = \frac{4}{ab} \int_{0}^{a} \left( \int_{0}^{b} u|_{t=0} \sin \left( \frac{\pi n y}{b} \right) \right) \sin \left( \frac{\pi m x}{a} \right) dx$$

In the same way, taking initial velocity $\frac{\partial u}{\partial t}$ at time $t = 0$ we get

$$B_{mn} = \frac{4}{ab \omega_{mn}} \int_{0}^{a} dx \int_{0}^{b} \frac{\partial u}{\partial t} \Bigg|_{t=0} \sin \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right).$$

8.6.3 Heat in a rectangular plate

Align a rectangular plate along the $x - y$ axes, so that it sits at $0 \leq x \leq a$ and $0 \leq y \leq b$. For heat in a rectangular plate, we go back through all of the stuff we did before, and find the heat equation

$$\frac{\partial u}{\partial t} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Now if we suppose that we use a thermostat to keep the edges at $0^\circ$, we derive in the usual way that the general solution is a sum of modes:

$$u(x, y, t) = \sum_{m, n=1}^{\infty} A_{mn} \sin \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right) \exp \left( -\omega_{mn}^2 t \right)$$

where

$$\omega_{mn} = \pi c \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2}. $$

This is similar to waves in a sheet. Again, we dig out the amplitudes $A_{mn}$ by setting $t = 0$ and find

$$A_{mn} = \frac{4}{ab} \int_{0}^{a} dx \int_{0}^{b} dy u(x, y, 0) \sin \left( \frac{\pi m x}{a} \right) \sin \left( \frac{\pi n y}{b} \right).$$

8.6.4 Steady states

In a wire, to get ends at different temperatures, we found the solution looked like $U(x, t) = s(x) + u(x, t)$ where $s(x)$ is the steady state, i.e. has the required
temperatures on the ends and doesn’t change in time, and \( u(x, t) \) is the solution of heat equation with zero temperatures at the ends.

In a rectangular plate, the same story:

\[
U(x, y, t) = s(x, y) + u(x, y, t)
\]

where \( s(x, y) \) is the steady state, and \( u(x, y, t) \) is a solution of the heat equation with zero temperature on the edges of the plate. BUT we can put a temperature function on each side of the plate, not just a number.

We already know the \( u(x, y, t) \) part: the general solution with zero edge temperatures. How do we find the steady state \( s(x, y) \)? It must satisfy the heat equation

\[
\frac{\partial s}{\partial t} = c^2 \left( \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} \right)
\]

and be time independent:

\[
\frac{\partial s}{\partial t} = 0.
\]

So it is a function satisfying

\[
\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = 0
\]

(the Laplace equation) which has to meet the required temperatures on the four edges:

\[
\begin{align*}
    s(x, 0) &= f_1(x) \\
    s(0, y) &= g_1(y) \\
    s(x, b) &= f_2(x) \\
    s(a, y) &= g_2(y)
\end{align*}
\]

where these \( f \) and \( g \) functions are the temperatures our thermostats are keeping the sides at.

Start with a simpler problem: suppose only the \( y = b \) side has a nonzero temperature profile, say \( f(x) \) as in figure \[34\] on the next page.

We start by looking for solutions of the Laplace equation which are products:

\[
s(x, y) = F(x)G(y).
\]

Plugging into the equation, we find

\[
\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}
\]

so both sides are constant, say \(-k\). Then since \( F(x) \) vanishes at both sides, \( F(0) = F(a) = 0 \), we must have \( k > 0 \) and

\[
F(x) = \sin \left( \frac{\pi n x}{a} \right)
\]
The requirement that the bottom \((y = 0)\) have temperature zero gives

\[ 0 = G(y) = A \]

so

\[ G(y) = B \sinh \left( \frac{\pi ny}{a} \right) \]

and the mode is

\[ s(x, y) = F(x)G(y) \]

\[ = B \sin \left( \frac{\pi nx}{a} \right) \sinh \left( \frac{\pi ny}{a} \right). \]

The general solution is a sum of modes

\[ s(x, y) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{\pi nx}{a} \right) \sinh \left( \frac{\pi ny}{a} \right). \]

How do we dig out the amplitudes? Look at the top side:

\[ f(x) = s(x, b) \]

\[ = \sum_{n=1}^{\infty} B_n \sin \left( \frac{\pi nx}{a} \right) \sinh \left( \frac{\pi nb}{a} \right). \]
This is just a Fourier series in $x$ so
\[ B_n = \frac{2}{a \sinh \left( \frac{\pi n b}{a} \right)} \int_0^a f(x) \sin \left( \frac{\pi n x}{a} \right) \, dx. \]

Now we can solve for heat in a plate, as long as three sides are kept at $0^\circ$ and the fourth at $f(x)$. Suppose we want to deal with general thermostats on the edges. Because we can add solutions to the heat equation (it is a linear equation), we get picture like figure 35 on the following page of adding together steady states. By swapping $x$ and $y$, or replacing $x$ with $a - x$, etc. you can figure out how to swap the sides of these boxes, so that we can apply the formula above for the steady state.

**Solved Problems #6**

1. Find the steady state of heat in a rectangular plate with edges all of length 1 kept at temperatures
- $f_1(x) = \sin 7\pi x$ bottom
- $f_2(x) = \sin \pi x$ top
- $g_1(y) = \sin 3\pi y$ left
- $g_2(y) = \sin 6\pi y$ right

You don’t have to compute a lot of integrals; instead just use the equation
\[ \int_0^1 \sin (\pi m x) \sin (\pi n x) \, dx = \begin{cases} \frac{1}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]
and the recipe on page 142. Your result should agree with the result of exercise 5 on page 143. Draw a plot of the steady state $u(x, y)$ in Maple and include it with your homework solutions.

**Solution:** The answer is given in the textbook:
\[
 f(x, y) = \frac{\sin (7\pi x) \sinh (7\pi (1 - y))}{\sinh (7\pi)} + \frac{\sin (\pi x) \sin (\pi y)}{\sinh (\pi)} + \frac{\sinh (3\pi (1 - x)) \sin (3\pi y)}{\sinh (3\pi)} + \frac{\sinh (6\pi x) \sin (6\pi y)}{\sinh (6\pi)}
\]

The picture is given in figure 36 on page 89. Notice how it looks roughly like a drum skin pulled tight.
Figure 35: Adding up steady states
Figure 36: A steady state: solving the Laplace equation
2. We would like to believe that a long skinny plate should lose heat more quickly than a squat, nearly square plate. What is the 1/2 life of the slowest decaying mode of the heat equation in a rectangular plate (say, with 0° at the edges)? Suppose that the plate has sides of length $a$ and $b = 1/a$, so total area 1. Show that the 1/2 life is longest (has a maximum—using calculus) for $a = 1$ (a square plate), and that the 1/2 life is near 0 for large $a$ or for $a$ near 0 (long skinny plates). You can use Maple to do the calculus, but include the Maple printout to show that you have $a = 1$ a strict maximum (i.e. derivative vanishing, and second derivative negative).

**Solution:** The 1/2 life of an exponentially decaying process

$$a_0e^{-\kappa t}$$

is the time $t$ when

$$e^{-\kappa t} = 1/2$$

which is easy to calculate:

$$t = \frac{\ln 2}{\kappa}.$$  

The modes of a rectangular plate look like

$$\exp\left(-\lambda_{mn}^2 t\right) \sin\left(\frac{\pi mx}{a}\right) \sin\left(\frac{\pi ny}{b}\right)$$

where

$$\lambda_{mn}^2 = \pi^2 c^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

so the exponential decay is just

$$\exp\left(-\lambda_{mn}^2 t\right)$$

and our $\kappa$ is just

$$\kappa = \lambda_{mn}^2 = \pi^2 c^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

and the slowest decaying mode is the one with the smallest value for this $\kappa$, i.e. the one with $m = n = 1$. Our 1/2 life is

$$t = \frac{\ln 2}{\kappa} = \frac{\ln 2}{\pi^2 c^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}$$

so for the slowest decaying mode it is

$$t = \frac{\ln 2}{\pi^2 c^2 \frac{1}{a^2} + \frac{1}{b^2}}.$$
Plugging in the fact that $b = 1/a$, we find
\[
t = \frac{\ln 2}{\pi^2 c^2} \frac{1}{a^2 + a^2}
\]
so that to find the rectangle with the longest $1/2$ life (and unit area) we have precisely to maximize this function $t(a)$ with respect to the choice of the side length $a$, or in other words to maximize the function
\[
\frac{1}{a^2 + a^2}
\]
This is the same as minimizing the function
\[
f(a) = \frac{1}{a^2} + a^2
\]
We find that the derivative of this function is
\[
f'(a) = 2 \left( a - \frac{1}{a^3} \right)
\]
which vanishes precisely when
\[
a = \frac{1}{a^3}
\]
or
\[
a^4 = 1.
\]
The value of $a$ is the length of a side of a rectangle, so it must be positive. Therefore,
\[
a = 1
\]
which is a unit square. This is only a critical point, not necessarily a maximum. The second derivative of our function $f(a)$ is
\[
f''(a) = 2 \left( 1 + \frac{3}{a^4} \right)
\]
which is positive for any positive $a$, so $a = 1$ is a local minimum of $f(a)$ and hence a local maximum of the $1/2$ life function $t(a)$. But there are no other critical points, and clearly when $a \to \infty$ or $a \to 0$, we find $t(a) \to 0$. Therefore the $t(a)$ function must actually be maximal at $a = 1$. The graph of the $1/2$ life as a function of side length $a$ is drawn in figure 37 on the following page. □
9 Changing coordinates: waves in a ball

9.1 The Laplacian in various coordinate systems

We run into the expression

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

a lot. We will write it

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and call it the Laplacian or Laplace operator.

$$\frac{\partial u}{\partial t} = \nabla^2 u \text{ heat equation}$$

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \text{ wave equation}$$

$$0 = \nabla^2 u \text{ Laplace (electrostatic potential) equation.}$$

In three dimensions, the Laplace operator is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$ 

In one dimension, it is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2}.$$ 

What does it mean? Consider in one dimension: it is just the second derivative, which is positive when the function $u$ is curving upward (bowl like) and negative...
when the function $u$ is curving downward, as in figure 38. So the heat equation says that a bowl-like function goes up (positive $\frac{\partial u}{\partial t}$), and a peak goes down—since $u$ represents temperature, it says that a hot spot cools, and a cold spot heats. The wave equation for a string says that a peak feels a force pulling it down, and a valley feels a force pulling it up—this is just the tension in the string.

For two and three dimensions, the Laplace operator $\nabla^2 u$ is just the sum of the second derivatives in each variable—essentially the average of how “bowl-like” the function is. So a bowl has positive Laplacian, a peak negative. For a saddle, see figure 39 on the following page.

Now we have to change to polar coordinates:

$$x = r \cos \theta$$
$$y = r \sin \theta.$$  

We start with

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and show that this equals

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$  

The proof is a long calculation given in the textbook. Let’s just check it on some examples. First, try it on $r^2$, since

$$r^2 = x^2 + y^2.$$
The Laplacian gives
\[ \nabla^2 r^2 = \nabla^2 (x^2 + y^2) \]
\[ = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} (x^2 + y^2) \]
\[ = 2 + 2 = 4. \]

In polar coordinates
\[ \nabla^2 r^2 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) r^2 \]
\[ = 2 + 2 = 4. \]

This is not very convincing, but if you try a few more functions, you will be convinced. For example, try
\[ x^k = r^k \cos(\theta)^k. \]

### 9.2 Vibrations of a circular membrane

Let \( u(r, \theta, t) \) be the height of the membrane (a drum head, for example) at time \( t \) and at position \( (x, y) = (r \cos \theta, r \sin \theta) \).

Then the wave equation is
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \]

Since the membrane is a circular disk, we use polar coordinates instead of \( (x, y) \) coordinates:
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right). \]
Our disk will have radius $a$ and will be tied down around the edge, i.e. at $r = a$ we set $u = 0$. We give it initial position

$$u(r, \theta, 0) = f(r, \theta)$$

and initial velocity

$$\frac{\partial u}{\partial t}(r, \theta, 0) = g(r, \theta).$$

Note that because $\theta$ represents an angle, $u(r, \theta, t)$ must be periodic in $\theta$ with period $2\pi$. To put this into the problem, we need

$$u(r, 0, t) = u(r, 2\pi, t)$$

and

$$\frac{\partial u}{\partial t}(r, 0, t) = \frac{\partial u}{\partial t}(r, 2\pi, t).$$

Let's first find the modes for this problem. Now we separate variables in polar coordinates:

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t)$$

a product of three functions. Plug in $u$ to the wave equation and get

$$R\Theta T'' = c^2 \left( R''\Theta T + \frac{1}{r} R'\Theta T + \frac{1}{r^2} R\Theta'' T \right).$$

Now divide both sides by $c^2 R\Theta T$ and get

$$\frac{T''(t)}{c^2 T(t)} = \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2 \Theta(\theta)}.$$

The left side has no $r$ or $\theta$ variables, and the right side no $t$ variable, so both sides are constant, say equal to $-\alpha$:

$$-\alpha = \frac{T''(t)}{c^2 T(t)}$$

$$-\lambda^2 = \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2 \Theta(\theta)}.$$

Now multiply the second equation by $r^2$:

$$-\alpha r^2 = r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} + r^2 \frac{\Theta''(\theta)}{\Theta(\theta)}$$

and separate the variables:

$$\alpha r^2 + \frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}.$$

Now the left side has only $r$ in it and the right side only $\theta$ in it, so both are constant, say equal to $\beta$.

$$0 = T''(t) + \alpha T(t)$$

$$0 = \Theta''(\theta) + \beta \Theta(\theta)$$

$$0 = r^2 R''(r) + r R'(r) + \left(\alpha r^2 - \beta\right) R.$$
Since the Θ function must repeat with period $2\pi$ (or any multiple of $2\pi$) we must have \( \beta \) positive spring constant and

\[
\Theta(\theta) = A \cos(m\theta) + B \sin(m\theta)
\]

and

\[
\beta = m^2.
\]

To avoid having the disk fly away over time, we can not allow the \( T \) function to have negative spring constant, or zero spring constant, so we must have \( \alpha > 0 \) too. But the hard part is the \( R(r) \) behaviour: pure radial motion of the wave. Like a rock dropped in water. It is not a sine or cosine function. In fact the ordinary differential equation

\[
0 = r^2 R''(r) + r R'(r) + \left( \alpha r^2 - m^2 \right) R
\]

with \( \alpha \) and \( k \) positive, and with the requirement \( R(a) = 0 \), has as solutions the functions

\[
R(r) = R_{mn}(r)
\]

where

\[
R_{mn}(r) = J_m(\lambda_{mn}r)
\]

with \( J_m \) the \( m \)-th Bessel function, and

\[
\lambda_{mn} = \frac{\alpha_{mn}}{a}
\]

where \( a \) is our radius and \( \alpha_{mn} \) is the \( n \)-th positive root of the \( m \)-th Bessel function \( J_m \). So it is a monster.

Note that as well as these, any sums like \( R(r) = 13R_{m.5}(r) + 1000R_{m.79}(r) \) is also a solution. But we will take the individual \( R_{mn} \) as our modes, and then form sums of modes.

The constant \( \alpha \) above turns out to be

\[
\alpha = \lambda_{mn}^2.
\]

This gives us the \( T \) function, since that solves an oscillator equation involving \( \alpha \):

\[
T(t) = A \cos(c\lambda_{mn}t) + B \sin(c\lambda_{mn}t).
\]

So now the modes of the wave equation in the disk are

\[
u(r, \theta, t) = \begin{cases} 
J_m(\lambda_{mn}r) \cos(m\theta) \cos(c\lambda_{mn}t) \\
J_m(\lambda_{mn}r) \cos(m\theta) \sin(c\lambda_{mn}t) \\
J_m(\lambda_{mn}r) \sin(m\theta) \cos(c\lambda_{mn}t) \\
J_m(\lambda_{mn}r) \sin(m\theta) \sin(c\lambda_{mn}t)
\end{cases}
\]
Adding them together, the general solution looks like

\[ u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[ J_m(\lambda_{mn} r) \left( a_{mn} \cos(m \theta) + b_{mn} \sin(m \theta) \right) \cos(c \lambda_{mn} t) 
+ (a^*_{mn} \cos(m \theta) + b^*_{mn} \sin(m \theta)) \sin(c \lambda_{mn} t) \right] \]

How do you find these \( a, b \) amplitudes from physical data? Clearly they are just Fourier series in \( \theta \), but in \( r \) they are horrible.

### 9.2.1 Bessel series

Any function \( f(r) \) for \( 0 \leq r \leq a \) which vanishes at \( r = a \) has an expansion

\[ f(r) = \sum_{j=1}^{\infty} A_j J_p(\lambda_{pj} r) \]

into Bessel functions, where

\[ \lambda_{pj} = \frac{\alpha_{pj}}{\alpha} \]

and \( \alpha_{pj} \) is the \( j \)-th zero of the \( p \)-th Bessel function \( J_p \). We can use any one \( J_p \) Bessel function we like to carry out such an expansion. To calculate the \( A_j \) amplitude:

\[ A_j = \frac{2}{a^2 J^2_{p+1}(\alpha_{pj})} \int_0^a f(r) J_p(\lambda_{pj} r) r dr. \]

These are just like Fourier series: we have modes, amplitudes, and functions written as sums of modes, and formulas for the amplitude of each mode.

### 9.2.2 Amplitudes of the wave equation on the disk

Putting it all together, we get the equations

\[ a_0 = \frac{1}{\pi a^2 J^2_0(\alpha_{0n})} \int_0^a \left( \int_0^{2\pi} f(r, \theta) \, d\theta \right) J_0(\lambda_{0n} r) \, r \, dr \]

\[ a_{mn} = \frac{2}{\pi a^2 J^2_{m+1}(\alpha_{mn})} \int_0^a \left( \int_0^{2\pi} f(r, \theta) \cos(m \theta) \, d\theta \right) J_m(\lambda_{mn} r) \, r \, dr \]

\[ b_{mn} = \frac{2}{\pi a^2 J^2_{m+1}(\alpha_{mn})} \int_0^a \left( \int_0^{2\pi} f(r, \theta) \sin(m \theta) \, d\theta \right) J_m(\lambda_{mn} r) \, r \, dr \]

\[ a^*_{0n} = \frac{1}{\pi \cos \alpha_0 a J^2_0(\alpha_{0n})} \int_0^a \left( \int_0^{2\pi} g(r, \theta) \, d\theta \right) J_0(\lambda_{0n} r) \, r \, dr \]

\[ a^*_{mn} = \frac{2}{\pi \cos \alpha_0 a J^2_{m+1}(\alpha_{mn})} \int_0^a \left( \int_0^{2\pi} g(r, \theta) \cos(m \theta) \, d\theta \right) J_m(\lambda_{mn} r) \, r \, dr \]

\[ b^*_{mn} = \frac{2}{\pi \cos \alpha_0 a J^2_{m+1}(\alpha_{mn})} \int_0^a \left( \int_0^{2\pi} g(r, \theta) \sin(m \theta) \, d\theta \right) J_m(\lambda_{mn} r) \, r \, dr. \]
9.3 Steady state temperature in the disk

Heat in a disk with edge fixed at 0°C is handled by the same methods we used for waves in a disk, and the general solution is

\[ u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \exp \left( -c^2 \lambda_{mn}^2 t \right) J_m(\lambda_{mn} r) \left( a_{mn} \cos (m\theta) + b_{mn} \sin (m\theta) \right) \]

and if the initial temperature is \( u(r, \theta, 0) = f(r, \theta) \) then

\[ a_{0n} = \frac{1}{\pi a^2 J_1'(\alpha_{0n})} \int_0^a \left( \int_0^{2\pi} f(r, \theta) \, d\theta \right) J_0(\lambda_{0n} r) \, r \, dr \]
\[ a_{mn} = \frac{2}{\pi a^2 J_{m+1}'(\alpha_{mn})} \int_0^a \left( \int_0^{2\pi} f(r, \theta) \cos(m\theta) \, d\theta \right) J_m(\lambda_{mn} r) \, r \, dr \]
\[ b_{mn} = \frac{2}{\pi a^2 J_{m+1}'(\alpha_{mn})} \int_0^a \left( \int_0^{2\pi} f(r, \theta) \sin(m\theta) \, d\theta \right) J_m(\lambda_{mn} r) \, r \, dr \]

As usual, we see exponentially decaying modes. Maple can handle the integrals.

But what if the edge is held at some other temperature, say by a thermostat? The steady states of heat in a disk are given by looking for time independent solutions \( s(r, \theta) \) to the heat equation

\[ \frac{\partial s}{\partial t} = c^2 \nabla^2 s \]

or in other words, solutions to the Laplace equation

\[ \nabla^2 s = 0. \]

First the modes: suppose \( s(r, \theta) = R(r)\Theta(\theta) \). Then plugging into the Laplace equation,

\[ 0 = \nabla^2 s = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) s \]

we find

\[ R'' \Theta + R' \frac{\Theta}{r} + R'' \Theta \frac{\Theta}{r^2} = 0 \]

and after separating variables,

\[ 0 = \Theta'' + \alpha \Theta \]
\[ 0 = r^2 R'' + r R' - \alpha R \]

Again, \( \Theta \) must have period \( 2\pi \) (or a multiple) in \( \theta \), since \( \theta \) represents an angle. So

\[ \Theta = A \cos n\theta + B \sin n\theta \]
and this forces $\alpha = n^2$ and
\[ r^2 R'' + rR' - n^2 R = 0. \]
This ordinary differential equation has a unique solution $r^n$ up to rescaling. It is convenient to rescale it to $(r/a)^n$. So the modes are
\[ s = \left( \frac{r}{a} \right)^n (A \cos(n\theta) + B \sin(n\theta)). \]
The general solution is a sum of modes:
\[ s(r, \theta) = \sum_{n=0}^{\infty} \left( \frac{r}{a} \right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)). \]
To find these $a$ and $b$ out of physical data, plug in $r = a$ to find the temperature at the edge of the disk:
\[ f(\theta) = s(a, \theta) = \sum_{n=0}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)). \]
This is exactly a Fourier series:
\[ a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta, \]
\[ a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) \, d\theta, \]
\[ b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) \, d\theta. \]

9.4 Bessel functions

Bessel’s equation of order $p$ is
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2) y = 0. \]
We will only consider it for $x > 0$, since in our applications $x$ will be replaced by some radius parameter $r$. The $p$ is a constant.
Let’s plug in a series
\[ y(x) = \sum_{n=0}^{\infty} a_n x^{q+n} \]
and see what solutions pop out. For such a function
\[ \frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (q+n)x^{q+n-1} \]
etc. So plugging into Bessel’s equation

\[ 0 = x^2 \sum_{n=0}^{\infty} a_n (q + n)(q + n - 1)x^{q+n-2} + \sum_{n=0}^{\infty} a_n (q + n)x^{q+n-1} + (x^2 - p^2) \sum_{n=0}^{\infty} a_n x^{q+n} \]

\[ = \sum_{n=0}^{\infty} a_n ((q + n)(q + n - 1) + (q + n) + x^2 - p^2)x^{q+n} \]

Let’s reorganize it, pulling out the bit with \(x^2\) in it, because it messes up the powers of \(x\):

\[ 0 = \sum_{n=0}^{\infty} a_n \left( (q + n)(q + n - 1) + (q + n)(1) - p^2 \right) x^{q+n} + \sum_{n=0}^{\infty} a_n x^{q+n+2} \]

\[ = \sum_{n=0}^{\infty} a_n \left( (q + n)^2 - p^2 \right) x^{q+n} + \sum_{n=2}^{\infty} a_{n-2} x^{q+n} \]

where in the last term, we shifted the index by 2. Let’s write down the various powers of \(x\), starting with the \(n = 0\) term:

\[ 0 = a_0 \left( (q + 0)^2 - p^2 \right) x^0 \]
\[ + a_1 \left( (q + 1)^2 - p^2 \right) x^{q+1} \]
\[ + a_2 \left( (q + 2)^2 - p^2 \right) x^{q+2} + a_0 x^{q+2} \]
\[ + a_3 \left( (q + 3)^2 - p^2 \right) x^{q+3} + a_1 x^{q+3} \]
\[ + \ldots \]

To get all of this to vanish, we need each term to vanish:

\[ 0 = a_0 (q^2 - p^2) \]
\[ 0 = a_1 ((q + 1)^2 - p^2) \]
\[ 0 = a_2 ((q + 2)^2 - p^2) + a_0 \]
\[ 0 = a_3 ((q + 3)^2 - p^2) + a_1 \]
\[ 0 = \ldots \]

Now since \(a_0\) is the coefficient of the lowest order term, if it vanished that wouldn’t be the lowest order term, so we can just suppose \(a_0 \neq 0\). This forces \(q = p\) or \(q = -p\). First let’s try \(q = p\).

Now with that settled, \(q^2 = p^2\) so the second equation forces \(a_1 = 0\), since \((q + 1)^2 > p^2\). Then looking at all of the rest of the terms, each \(a_n\) is some multiple of \(a_{n-2}\), two steps back. So \(a_1 = 0\) forces \(0 = a_3 = a_5 = \ldots\). We only need \(a_2, a_4, \ldots\). The equations above give

\[ 0 = a_{2k+2} \left( (p + (2k + 2))^2 - p^2 \right) + a_{2k} \]
so that

\[
a_{2k+2} = - \frac{a_{2k}}{(p + 2k + 2)^2 - p^2} = - \frac{a_{2k}}{4(k + 1)(k + p + 1)}
\]

Try out some examples:

\[
a_2 = - \frac{a_0}{4(1)(p + 1)} = - \frac{a_0}{4(p + 1)}
\]

\[
a_4 = - \frac{a_2}{8(2)(p + 2)} = - \frac{a_2}{8(p + 2)}
\]

\[
= - \frac{a_0}{8(p + 1)}
\]

\[
= \frac{a_0}{4^2(p + 1)(p + 2)}
\]

\[
a_6 = - \frac{a_4}{4(3)(p + 3)}
\]

\[
= - \frac{a_0}{4^3(3)(p + 3)(p + 2)(p + 1)}.
\]

If we see the pattern, we find

\[
a_{2k} = \frac{(-1)^k a_0}{4^k k!(p + k)(p + (k - 1)) \ldots (p + 3)(p + 2)(p + 1)}.
\]

To simplify this a bit, define the Gamma function \( \Gamma(x) \) to be

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.
\]

This makes sense as long as \( x > 0 \) because then the integral is convergent: for large \( t \) it looks like exponential decay, and so converges rapidly, while for small \( t \) it looks like a power of \( t \) and for \( x > 0 \) that integral converges. Easy integration by parts gives the \textit{functional equation}:

\[
\Gamma(x + 1) = x \Gamma(x).
\]

The one integral that you can easily do is \( \Gamma(1) = \int e^{-t} \, dt = 1 \). From here apply the functional equation to get \( \Gamma(2) = 1 \cdot \Gamma(1) = 1 \), and \( \Gamma(3) = 2 \Gamma(2) = 2 \) etc. so that

\[
\Gamma(n + 1) = n!
\]

for \( n \) nonnegative integer.

Using the functional equation, we find

\[
\Gamma(x) = \frac{1}{x} \Gamma(x + 1)
\]
which we use to define $\Gamma(x)$ for $x < 0$. For example,

$$\Gamma(-1/2) = \frac{1}{-1/2} \Gamma(-1/2 + 1) = -2 \Gamma(1/2).$$

This doesn’t work for defining $\Gamma(n)$ for $n$ a negative integer, since $\Gamma(0)$ is not defined. In fact, $\Gamma(x)$ is infinite at $x = 0, -1, -2, \ldots$. Using the functional equation over and over we get

$$\Gamma(p + k + 1) = (p + k)\Gamma(p + k) = (p + k)(p + k - 1)\Gamma(p + k - 1) = \ldots = (p + k)(p + k - 1) \ldots (p + 1)\Gamma(p + 1).$$

Therefore our coefficients $a_{2k}$ are

$$a_{2k} = \frac{(-1)^k a_0}{4^k k! \Gamma(p + k + 1) / \Gamma(p + 1)}.$$

Now we use the last bit of freedom left: we can pick $a_0$ to be anything we like, because a rescaling of a solution is also a solution. We pick

$$a_0 = \frac{1}{2^p \Gamma(p + 1)}$$

so that

$$a_{2k} = \frac{(-1)^k}{4^k k! \Gamma(p + k + 1)}$$

or we can write this as

$$a_{2k} = \frac{(-1)^k}{2^{2k} k! \Gamma(p + k + 1)}$$

and our the solution $J_p(x)$ to the Bessel equation is

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(p + k + 1)} \left( \frac{x}{2} \right)^{2k+p}.$$

Now we can return to our earlier choice of $q = p$ instead of $q = -p$. We picked $q = p$, but could do exactly the same with $q = -p$ and we get another solution: $J_{-p}(x)$. But they might be related. In fact, they are: it turns out that for $p = n$ an integer ($n = 0, \pm 1, \pm 2$, etc.)

$$J_{-n}(x) = (-1)^n J_n(x).$$

So this is not a new solution—just the old one rescaled. But for $p$ not an integer, it is actually a new solution, so all of the solutions are just $aJ_p(x) + bJ_{-p}(x)$.  

102
To find the extra solution over the integers, we define

\[ Y_p(x) = \frac{J_p(x) \cos(\pi p) - J_{-p}(x)}{\sin(\pi p)}. \]

This is just a combination of \( J_p(x) \) and \( J_{-p}(x) \) so it solves Bessel’s equation. But now we take the limit as \( p \) approaches an integer value. It turns out that \( Y_n(x) \) is actually an independent solution of the Bessel equation of integer order \( n \). So for any \( p \), even integer \( p = n \), we have the complete collection of solutions: \( aJ_p(x) + bY_p(x) \).

### 9.5 Bessel series

Now that we have the ability to calculate Bessel functions order by order in Taylor expansions, we can teach a computer how to calculate them (we won’t actually do this—our computers already know how), and we want to understand how to write a function in terms of a series of Bessel functions, just like Fourier series.

Suppose that we can write any function \( f(x) \) as a sum of Bessel functions like

\[ f(x) = \sum_{n=0}^{\infty} A_n J_p(\lambda_{pn}x) \]

where

\[ \lambda_{pn} = \frac{\alpha_{pn}}{a} \]

and \( \alpha_{pn} \) is the \( n \)-th root of \( J_p(x) \). We will always suppose this, and never prove it. But we will prove: we can calculate these \( A_n \) using the formula

\[ A_n = \frac{2}{a^2 J_{p+1}(\alpha_{pn})} \int_0^a f(x)J_p(\lambda_{pn}x) x \, dx. \]

First, we rewrite the Bessel equation

\[ x^2y'' + xy' + (x^2 - p^2)y = 0 \]

as

\[ \frac{d}{dx}(xy') = -\frac{x^2 - p^2}{x}y. \]

Now define functions

\[ \phi_n(x) = J_p(\lambda_{pn}x). \]

In particular

\[ \phi_n(a) = J_p(\alpha_{pn}) = 0. \]

We can rewrite Bessel’s equation once again, this time as

\[ \frac{d}{dx} \left( x \frac{d\phi_n}{dx} \right) = -\frac{\lambda_{pn}^2 x^2 - p^2}{x} \phi_n \]
which is easy to check. Now if we have two different $\phi$ functions, say $\phi_n$ and $\phi_m$ we get, subtracting the two corresponding Bessel equations,

$$\phi_m \frac{d}{dx} \left( x \frac{d\phi_n}{dx} \right) - \phi_n \frac{d}{dx} \left( x \frac{d\phi_m}{dx} \right) = \left( \lambda_m^2 - \lambda_n^2 \right) \phi_m(x) \phi_n(x)x.$$

The left hand side is

$$\phi_m \frac{d}{dx} \left( x \frac{d\phi_n}{dx} \right) - \phi_n \frac{d}{dx} \left( x \frac{d\phi_m}{dx} \right) = \frac{d}{dx} \left( x\phi_m \frac{d\phi_n}{dx} - x\phi_n \frac{d\phi_m}{dx} \right)$$

so putting that together

$$\frac{d}{dx} \left( x\phi_m \frac{d\phi_n}{dx} - x\phi_n \frac{d\phi_m}{dx} \right) = \left( \lambda_m^2 - \lambda_n^2 \right) x\phi_m \phi_n.$$

if we integrate both sides

$$x\phi_m \frac{d\phi_n}{dx} - x\phi_n \frac{d\phi_m}{dx} \bigg|_{x=0}^{x=a} = \int_0^a \left( \lambda_m^2 - \lambda_n^2 \right) \phi_m \phi_n x \, dx.$$

The left hand side is

$$a\phi_m(a) \frac{d\phi_n}{dx}(a) - a\phi_n(a) \frac{d\phi_m}{dx}(a) = 0$$

because $\phi_m(a) = \phi_n(a) = 0$. Therefore

$$\left( \lambda_m^2 - \lambda_n^2 \right) \int \phi_m(x) \phi_n(x) x \, dx = 0.$$

Dividing by $\lambda_m^2 - \lambda_n^2$ we get

$$\int \phi_m(x) \phi_n(x) x \, dx = 0.$$

Now we want to calculate this integral when $m = n$. This is as far as we are going to go—the rest is about as complicated. Let us just state the result:

$$\int_0^a \phi_n(x) x \, dx = \frac{a^2 J_{p+1}(\alpha_{pn})}{2}.$$

Putting this together, we can say that if

$$f(x) = \sum A_n \phi_n(x)$$

then we can dig out the $A_n$ as follows:

$$\int f(x) \phi_m(x) x \, dx = \int \sum_n A_n \phi_n(x) \phi_m(x) x \, dx$$

$$= \sum A_n \int \phi_n(x) \phi_m(x) x \, dx$$

$$= A_m \frac{a^2 J_{p+1}(\alpha_{pn})}{2}.$$
So to pick out $A_m$, compute the integral

$$A_m = \frac{2}{a^2 f_{p+1}^2(\alpha_{pm})} \int_0^a f(x) \phi_m(x) x \, dx.$$ 

## 10 Hanging chains and elastic beams

### 10.1 The hanging chain

We draw a hanging chain in a funny way: picture a function $u(x)$ as the chain, but with gravity pulling to the left, instead of down, and the chain tied on at $x = L, u = 0$ (at the right), as in figure [40]. This is so that as it wiggles near equilibrium, it is always a function of $x$ for $0 \leq x \leq L$.

This is a lot like the wave equation. The tension pulling on a point of the chain is $\tau = \rho g x$ where $g$ is gravitational acceleration, and $\rho$ is the density—this says that far up the string, you have the whole weight of the string underneath you pulling, while near the bottom, you have very little string underneath you pulling. The tension pulls the string in the tangent direction, so the force on little piece of string between $x$ and $x + \Delta x$ is

$$F = \tau(x + \Delta x) \sin \theta(x + \Delta x) - \tau(x) \sin \theta(x)$$

where $\theta(x)$ is the direction of the tangent to the chain. Mass of this piece is approximately $\rho \Delta x$, so acceleration is $a = \frac{\partial^2 u}{\partial t^2} = F/m$ giving

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho} \frac{\tau(x + \Delta x) \sin \theta(x + \Delta x) - \tau(x) \sin \theta(x)}{\Delta x}.$$
Now use the equation for tension, \( \tau(x) = \rho g x \) and the approximation

\[
\sin \theta(x) \sim \tan \theta(x) = \frac{\partial u}{\partial x}
\]

for small angles \( \theta(x) \). This gives

\[
\frac{\partial^2 u}{\partial t^2} = \frac{g(x + \Delta x) \frac{\partial u}{\partial x}(x + \Delta x) - g x \frac{\partial u}{\partial x}(x)}{\Delta x}
\]

Take the limit as \( \Delta x \to 0 \):

\[
\frac{\partial^2 u}{\partial t^2} = g \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right)
\]

which we can also write as

\[
\frac{\partial^2 u}{\partial t^2} = g \frac{\partial u}{\partial x} + g x \frac{\partial^2 u}{\partial x^2}
\]

the equation of a hanging chain.

Let's look for modes. Suppose \( u(x, t) = X(x)T(t) \). Then the equation of a hanging chain gives

\[
XT'' = gT(xX'' + X').
\]

Separate variables to get

\[
0 = T'' + \alpha g T
\]

\[
0 = xX'' + X' + \alpha X
\]

The first equation is a harmonic oscillator in time \( t \), so

\[
T(t) = A \cos(\omega t) + B \sin(\omega t)
\]

where the frequency \( \omega \) is \( \omega = \sqrt{\alpha g} \). Then we have to solve the equation

\[
0 = xX''(x) + X'(x) + \alpha X(x)
\]

with \( \alpha > 0 \) a constant. Rewrite it as

\[
0 = \frac{d}{dx} \left( xX'(x) \right) + \alpha X(x).
\]

We will not prove it, but with the change of variable \( z = 2\sqrt{x} \) we can get this to become a Bessel equation, and the solutions are

\[
X(x) = J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right)
\]

where \( n = 1, 2, 3, \ldots \) and \( \alpha_n \) is the \( n \)-th positive root of \( J_0 \), and the constant \( \alpha \) turns out to be

\[
\alpha = \frac{\alpha_n}{4\pi} \sqrt{gL}.
\]
So the modes are
\[ X(x)T(t) = J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) (A \cos (\omega_n t) + B \sin (\omega_n t)) \]
where
\[ \omega_n = \frac{\alpha_n}{2} \sqrt{\frac{g}{L}}. \]

Putting together the modes, the general solution is
\[ u(x, t) = \sum_{n=1}^{\infty} J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) (A_n \cos (\omega_n t) + B_n \sin (\omega_n t)). \]

Let \( f(x) \) be the initial position of the string: \( u(x, 0) = f(x) \) and \( g(x) \) be the initial velocity:
\[ \frac{\partial u}{\partial t} (x, 0) = g(x). \]

Putting in \( t = 0 \) we get
\[ u(x, 0) = \sum_{n=1}^{\infty} A_n J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \]
something like a Bessel series; we won’t prove it, but in fact
\[ A_n = \frac{1}{LJ_1 (\alpha_n)^2} \int_0^L f(x) J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \, dx. \]

Similarly, the \( B_n \) are given in terms of initial velocity:
\[ B_n = \frac{2}{\alpha_n \sqrt{gLJ_1 (\alpha_n)^2}} \int_0^L g(x) J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \, dx. \]

**Solved Problems #7**

1. On the web page, you will find a Maple worksheet about hanging chains. The center of mass of a hanging chain described by a function \( u(x, t) \) is
\[ (x_{\text{center}}, u_{\text{center}}) = \left( \frac{L}{2}, \frac{1}{L} \int_0^L u \, dx \right) \]
Using this worksheet, explain why the chain of length \( L = 1 \) with
- initial velocity \( v(x) = 0 \)
- initial position \( f(x) = L - x \)
has center of mass oscillating close to (but not quite equal to) a cosine wave, while with
- initial velocity \( v(x) = 0 \)
- initial position \( f(x) = x(L - x) \)
Figure 41: The hanging chain amplitudes. Notice that when \( f(x) = L - x \) the first amplitude clearly dominates, while with \( f(x) = x(L - x) \) there are more nonzero amplitudes.

it has center of mass oscillating rather wildly, with no obvious pattern. Hint: look at the amplitudes, or the pictures on page 304 of the textbook. Also, you only have to change the function \( f(x) \) in the Maple worksheet, and use Edit---Execute---Worksheet to see how these chains look.

**Solution:** Because the initial velocities \( v(x) \) in each example are zero, the amplitudes \( B_j \) all vanish. Therefore we can concentrate on the \( A_j \), and our chain displacement function is

\[
u(x,t) = \sum_{j=1}^{\infty} A_j J_0 \left( \alpha_j \sqrt{\frac{x}{L}} \right) \cos \left( \sqrt{\frac{g}{L}} \frac{\alpha_j^2}{2} t \right),
\]

a sum of modes of higher and higher frequency oscillations.

From the pictures on page 304, you can easily see that the first mode of the hanging chain, the function \( J_0 (\alpha_1 \sqrt{x}) \) is very close to a linear function. Therefore, the linear function

\[ f(x) = L - x \]

is very well modelled by the first mode, and so it has a large first amplitude \( A_1 \). This is clear from the plot of the amplitudes (see figure 41).

On the other hand, the function

\[ f(x) = x(L - x) \]
(a) The initial position of the first chain

(b) The center of mass of the first chain

Figure 42: The first chain

does not look like any of the modes. You can see this clearly with the first three modes, by looking at the pictures from the textbook. And the higher frequency modes not pictured there have even more “bumps” on them (they have more complicated pictures). It is even clearer when looking at the amplitudes in figure 41 on the page before.

To see how this affects the center of mass, note that

\[ u_{\text{center}} = \frac{1}{L} \int_0^L u \, dx \]

\[ = \frac{1}{L} \int_0^L \sum_{j=1}^\infty A_j J_0 \left( \frac{\alpha_j}{L} \sqrt{\frac{x}{L}} \right) \cos \left( \sqrt{\frac{g}{L} \frac{\alpha_j^2}{2}} t \right) \]

\[ = \sum_{j=1}^\infty A_j \left( \frac{1}{L} \int_0^L J_0 \left( \frac{\alpha_j}{L} \sqrt{\frac{x}{L}} \right) \, dx \right) \cos \left( \sqrt{\frac{g}{L} \frac{\alpha_j^2}{2}} t \right) \]

which is a Fourier series in cosines in time \( t \). In the first case, if \( f(x) = L - x \), then the Fourier series consists largely of just the first term, since \( A_2, A_3, \ldots \) are close to zero. This first term has a single cosine in time, which dominates the center of mass, as in figure 42.

The same sort of Fourier series emerges for the second chain, but it has a lot of large contributions from various amplitudes \( A_j \) (especially the first two), and so it is a complicated mixture of cosines at different frequencies. This accounts for its more complicated behaviour, as seen in figure 43 on the next page.

\[ \square \]
(a) The initial position of the second chain

(b) The center of mass of the second chain

Figure 43: The second chain

(a) The first mode of the hanging chain

(b) The second mode of the hanging chain

Figure 44:
10.2 The vibrating beam

Consider a long thin solid beam vibrating. It is possible to show (we won’t do it) that its height $u$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}$$

where $c$ is a physical constant depending on the shape of the beam and its material. Looking for modes $u = X(x)T(t)$ we find

$$X(x) \frac{d^2 T}{dt^2} = -c^2 \frac{d^4 X}{dx^4} T(t)$$

and separating variables gives

$$\frac{1}{c^2 T(t)} \frac{d^2 T}{dt^2} = -\frac{1}{X(x)} \frac{d^4 X}{dx^4}$$

and thus both sides are constant, say equal to $-k$.

$$0 = \frac{d^2 T}{dt^2} + c^2 k T(t)$$

$$0 = \frac{d^4 X}{dx^4} - k X(x).$$

The first equation is an oscillator, and since we can’t let the beam fly away, the solutions must be sines and cosines:

$$T(t) = A \cos(\omega t) + B \sin(\omega t)$$
Figure 46: The beam on the left has a rigidly fastened left end, and a free right end. The beam on the right has both ends simply fastened

where $\omega = c\sqrt{k}$. The solution of the $X$ equation is

$$X(x) = C \cos \left( k^{1/4}x \right) + D \sin \left( k^{1/4}x \right) + E \cosh \left( k^{1/4}x \right) + F \sinh \left( k^{1/4}x \right)$$

Now we have to impose some conditions on the ends of the beam. See figure 46:

Rigidly fastened: $u = \frac{\partial u}{\partial x} = 0$

Free: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0$

Simply fastened: $u = \frac{\partial^2 u}{\partial x^2} = 0$.

For simplicity, let’s take both ends simply fastened. Suppose its length is $L$. Then we must have

$$u = \frac{\partial^2 u}{\partial x^2} = 0$$

at both ends, $x = 0$ and $x = L$. Applying this to our mode, we must have $X = dX/dx = 0$ at both ends. Plugging this in, we find that at $x = 0$ this forces

$$0 = X(0) = C + E$$

$$0 = \frac{d^2 X}{dx^2}(0) = -C + E$$

and to have both vanish forces $C = E = 0$. At the other end, the same conditions force $F = 0$ and force the frequency to be $k^{1/4} = \pi n/L$ for an integer $n$. So up to rescaling (which we absorb into $T(t)$) we get

$$X(x) = \sin \left( \frac{\pi nx}{L} \right)$$

and we get

$$k = \left( \frac{\pi n}{L} \right)^4$$

112
so that the mode is

\[ u = X(x)T(t) = \sin (p_n x) \left( A \cos (\omega_n t) + B \sin (\omega_n t) \right) \]

where

\[ p_n = \frac{\pi n}{L} \]
\[ \omega_n = c p_n^2 \]

Putting together the modes, the general solution is

\[ u(x, t) = \sum_{n=1}^{\infty} \sin (p_n x) \left( A_n \cos (\omega_n t) + B_n \sin (\omega_n t) \right) . \]

If we know the initial position \( f(x) = u(x, 0) \) and initial velocity \( g(x) = \frac{\partial u}{\partial t} (x, 0) \) then we can plug in \( t = 0 \) and find

\[ f(x) = u(x, 0) = \sum_{n=1}^{\infty} \sin (p_n x) A_n = \sum_{n=1}^{\infty} A_n \sin \left( \frac{\pi n x}{L} \right) \]

a Fourier sine series, so

\[ A_n = \frac{2}{L} \int_0^L f(x) \sin (p_n x) \, dx. \]

Similarly

\[ B_n = \frac{2}{L \omega_n} \int_0^L g(x) \sin (p_n x) \, dx. \]

What physics do we see? The solution looks a lot like the solution of the wave equation, with sines and cosines in space and time, but the frequencies are related differently. For the wave equation, the spatial frequency (in \( x \)) is \( p_n = \pi n/L \) and the time frequency (of vibration, i.e. in \( t \)) is \( \omega_n = \pi nc/L = cp_n \). But for this beam, the spatial frequency is \( p_n = \pi n/L \) still the same, while the frequency in time is \( \omega_n = cp_n^2 \). So high frequencies of a string, when you make it rigid and turn it into a beam, become very high. For instance, if the string is vibrating at 1000 cycles per second, then a rigid beam with the same initial shape might vibrate at 1000000 cycles per second. (This assumes that the constants \( c \) are the same in the string and the beam—keep in mind that these constants are unrelated in general).

10.3 The beam with rigidly fixed ends

Suppose that both ends are rigidly fixed (also called clamped). Then we go back to our modes and impose the conditions

\[ u = \frac{\partial u}{\partial x} = 0 \]
on the ends instead. Recall our function $X(x)$ was

$$X(x) = C \cos \left( k^{1/4} x \right) + D \sin \left( k^{1/4} x \right) + E \cosh \left( k^{1/4} x \right) + F \sinh \left( k^{1/4} x \right).$$

Imposing the condition $u = \frac{\partial u}{\partial x} = 0$ at $x = 0$ gives

$$C = -E$$

and

$$D = -F.$$ 

But the conditions at the other end, $u = \frac{\partial u}{\partial x} = 0$ at $x = L$ give horrible equations:

$$0 = C \cos \left( k^{1/4} L \right) + D \sin \left( k^{1/4} L \right) + E \cosh \left( k^{1/4} L \right) + F \sinh \left( k^{1/4} L \right)$$

$$0 = -k^{1/4} C \sin \left( k^{1/4} L \right) + k^{1/4} D \cos \left( k^{1/4} L \right) + k^{1/4} E \sinh \left( k^{1/4} L \right) + k^{1/4} F \cosh \left( k^{1/4} L \right).$$

We can divide off a $k^{1/4}$ from the second of these equations, and plug in $C = -E$ and $D = -F$ to get

$$C = -E$$

and

$$D = -F$$

$$0 = (\cosh(r) - \cos(r)) E + (\sinh(r) - \sin(r)) F$$

$$0 = (\sinh(r) + \sin(r)) E + (\cosh(r) - \cos(r)) F$$

where $r = k^{1/4} L$. So in matrix form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh(r) - \cos(r) & \sinh(r) - \sin(r) \\ \sinh(r) + \sin(r) & \cosh(r) - \cos(r) \end{pmatrix} \begin{pmatrix} E \\ F \end{pmatrix}.$$

If these two linear equations on the two variables $E$ and $F$ are independent, then there is only one solution—but that would have to be $E = F = 0$, because that is a solution. But then the whole mode would be zero.
So to have any modes at all, the two equations can not be independent, so that matrix must have determinant zero:

\[
0 = \det \begin{pmatrix}
cosh(r) - \cos(r) & \sinh(r) - \sin(r) \\
\sinh(r) + \sin(r) & \cosh(r) - \cos(r)
\end{pmatrix}.
\]

This determinant, after a little simplifying, turns out to be

\[
2 - 2 \cosh(r) \cos(r).
\]

This vanishes just exactly when

\[
\cos(r) = \frac{1}{\cosh(r)}.
\]

When that happens, we can solve for \(C, D, E\) in terms of \(F\):

\[
C = -E \\
D = -F \\
F = -E \frac{\cosh(r) - \cos(r)}{\sinh(r) - \sin(r)}.
\]

We can rescale, so take \(E = 1\). This gives

\[
C = -1 \\
D = \frac{\cosh(r) - \cos(r)}{\sinh(r) - \sin(r)} \\
E = 1 \\
F = -\frac{\cosh(r) - \cos(r)}{\sinh(r) - \sin(r)}.
\]

We still need to find the possibilities for \(r\), solving the equation

\[
\cos(r) = \frac{1}{\cosh(r)}.
\]

This is too horrible to be solved by hand. We have to find all of the \(r\) that satisfy this, and use them as \(k = (r/L)^4\) in our spring constants for the \(T(t)\) equation.

Suppose that we let \(r_n\) be the positive roots of the equation

\[
\cos(r_n) = \frac{1}{\cosh(r_n)}.
\]

Then we have modes:

\[
u(x, t) = X_n(x) \left( A \cos(\omega_n t) + B \sin(\omega_n t) \right)
\]
Figure 48: Modes $X_1(x), X_2(x), \ldots$ of a rigidly fastened beam. They look like sine waves, just like for the wave equation, except that they tail off at either end so that they are tangent to the $x$-axis.

where

$$X_n(x) = \cosh \left( \frac{r_n x}{L} \right) - \cos \left( \frac{r_n x}{L} \right) - \cosh \left( r_n \right) - \cos \left( r_n \right) \left( \sinh \left( \frac{r_n x}{L} \right) - \sin \left( \frac{r_n x}{L} \right) \right)$$

and

$$\omega_n = c \left( \frac{r_n}{L} \right)^2.$$

The first few $r_n$ are

- $r_1 = 4.730040745$
- $r_2 = 7.853204624$
- $r_3 = 10.99560784$
- $r_4 = 14.13716549$
- $r_5 = 17.27875966$
- $r_6 = 20.42035225$
- $r_7 = 23.56194490$

Maple code to find all of these $r_n$ is on the web page.

The general solution is a sum of modes:

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) \left( A_n \cos (\omega_n t) + B_n \sin (\omega_n t) \right).$$

Once again we have to find these $A_n$ and $B_n$ out of the initial position $f(x) = u(x,0)$ and initial velocity $g(x) = \frac{\partial u}{\partial t}(x,0).$ General Sturm–Liouville
theory, which we will not cover, gives the answers: first let
\[ \kappa_n = \int_0^L X_n(x)^2 \, dx \]
and then
\[ A_n = \frac{1}{\kappa_n} \int_0^L f(x) X_n(x) \, dx \]
\[ B_n = \frac{L^2}{\kappa_n c r_n^2} \int_0^L g(x) X_n(x) \, dx. \]
See figure 49 on the next page to compare how different the behaviour is from the simply fastened beam.

What physics do we see? The low frequencies are very hard to uncover: we use Maple. For large \( r \) the solutions of \( \cos(r) \cosh(r) = 1 \) are approximately close to the zeros of \( \cos(r) \) because the \( \cosh(r) \) is growing exponentially rapidly. So these values \( r_n \) are approximately \( r_n \sim \pi/2 + \pi n \) for large \( n \). This makes the frequencies in space be roughly
\[ p_n = \frac{r_n}{L} \]
while in time
\[ \omega_n = c (r_n/L)^2 = cp_n^2 \sim c (\pi/2 + \pi n)^2 / L^2 \]
roughly the same as we saw for the simply fixed beam. Again, they are typically very fast, growing quadratically, compared to string vibrations, which are like \( \omega_n = c \pi n / L \) growing linearly in \( n \).

11 The Fourier transform: waves in infinite space

We have looked at waves on strings, in rectangular sheets, and in disks. We could look at boxes, cylinders, and spheres using the same ideas. What about fields spread out in infinite space?

Recall the string. If the ends are miles away, then you don’t see them anymore, and you can think that there are no ends. Previously, we took a piece of string, and pretended it was periodic, and then took Fourier series. Now we wonder what happens to Fourier theory if the period gets “infinite;” in other words there is no period.

Recall that we expressed periodic functions as
\[ f(x) = a_0 + \sum_{n=0}^{\infty} (a_n \cos(\omega_n x) + b_n \sin(\omega_n x)) \]
where the frequencies are
\[ \omega_n = 2\pi n / T \]
Figure 49: Two beams with the same initial position, and zero initial velocity, differing only in how the beams are fastened at the ends.

with $T$ the period. So the frequencies are spaced apart by an amount

$$\Delta \omega = 2\pi/T.$$ 

Now if $T = \infty$ then $\Delta \omega$ is an infinitely small step, and instead of just adding up these discretely spaced frequencies we have to add up all frequencies:

$$f(x) = \int_{-\infty}^{\infty} (A(\omega) \cos(\omega x) + B(\omega) \sin(\omega x)) \, d\omega.$$

No funny business about $a_0$ anymore. To dig out the $A$ and $B$ amplitudes we take the old formulas

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos(\omega_n x) \, dx = \frac{\Delta \omega}{\pi} \int_{-T/2}^{T/2} f(x) \cos(\omega_n x) \, dx$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin(\omega_n x) \, dx = \frac{\Delta \omega}{\pi} \int_{-T/2}^{T/2} f(x) \cos(\omega_n x) \, dx$$

and make the analogous formulas:

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\omega x) \, dx$$
If we use complex exponentials, then the formulas are:

\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\omega) e^{i\omega x} \, d\omega \]  \hspace{1cm} (1)

and

\[ C(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx. \]

This \( C(\omega) \) is usually the nicest to work with, and is called the Fourier transform of \( f(x) \), and is written \( C(\omega) = \hat{f}(\omega) \) or \( C(\omega) = \mathcal{F}(f)(\omega) \). Formula (1) to calculate \( f(x) \) in terms of \( C(\omega) \) is called the inverse Fourier transform.

We call \( A(\omega) \) the Fourier cosine transform, and \( B(\omega) \) the Fourier sine transform.

**Solved Problems #8**

1. Compute the Fourier cosine and sine transforms of

\[ f(x) = \begin{cases} 
 1 & \text{if } -1 < x < 1 \\
 0 & \text{otherwise}. 
\end{cases} \]

**Solution:**

\[
A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx \\
= \frac{1}{\pi} \int_{-1}^{1} \cos(\omega x) \, dx \\
= \frac{1}{\pi} \left[ \frac{\sin(\omega x)}{\omega} \right]_{x=-1}^{x=1} \\
= \frac{1}{\pi} \left( \frac{\sin(\omega)}{\omega} - \frac{-\sin(-\omega)}{\omega} \right) \\
= \frac{2\sin(\omega)}{\pi \omega}.
\]

Because \( f(x) \) is even, \( B(\omega) = 0 \). See figure 50 on the next page. \( \square \)

2. Take \( a \) to be a positive constant. Find the Fourier transforms of the functions:

(a)

\[ f(x) = \begin{cases} 
 1/a & \text{if } -1/a \leq x \leq 1/a \\
 0 & \text{otherwise} 
\end{cases} \]
(a) A square wave

(b) The Fourier cosine transform of the square wave

Figure 50: A function and its Fourier cosine transform—the Fourier sine transform is zero

Solution:

\[ \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-1/a}^{1/a} ae^{-i\omega x} \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \begin{cases} a & \text{if } \omega \neq 0 \\ 2 & \text{if } \omega = 0 \end{cases} \left( e^{-i\omega/a} \left|_{x=1/a} \right. \right. \left. \left. - e^{-i\omega/a} \left|_{x=-1/a} \right. \right. \right. \]

\[ = \frac{1}{\sqrt{2\pi}} \begin{cases} 2 & \text{if } \omega \neq 0 \\ 2 & \text{if } \omega = 0 \end{cases} \left( e^{i\omega/a} - e^{-i\omega/a} \right) \]

\[ = \frac{1}{\sqrt{2\pi}} \begin{cases} 2 & \text{if } \omega \neq 0 \\ 2 & \text{if } \omega = 0 \end{cases} \left( \frac{\sin(\omega/a)}{\omega/a} \right) \]

\[ = \frac{2}{\sqrt{2\pi}} \begin{cases} 1 & \text{if } \omega \neq 0 \\ 1 & \text{if } \omega = 0 \end{cases} \]
(b) 

\[ f(x) = ae^{-a|x|} \]

Hint: split into integrals over positive and over negative \( x \).

Solution:

\[ \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ae^{-a|x|} e^{-i\omega x} \, dx \]

\[ = \frac{a}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{ax} e^{-i\omega x} \, dx + \int_{0}^{\infty} e^{-ax} e^{-i\omega x} \, dx \right) \]

\[ = \frac{a}{\sqrt{2\pi}} \left( \int_{-\infty}^{0} e^{(a-i\omega)x} \, dx + \int_{0}^{\infty} e^{-(a+i\omega)x} \, dx \right) \]

At this point we need to look at how these expressions behave near \( -\infty \) and \( \infty \). For example,

\[ e^{(a-i\omega)x} = e^{ax} e^{-i\omega x} \]

\[ = e^{ax} (\cos \omega x + i \sin \omega x) \]

Since the \( e^{ax} \) term is exponential growth, it dies off at \( x = -\infty \), while the \( \cos \omega x \) and \( \sin \omega x \) terms remain bounded. Therefore

\[ e^{(a-i\omega)x} \to 0 \text{ as } x \to -\infty. \]

Similarly,

\[ e^{-(a+i\omega)x} \to 0 \text{ as } x \to \infty. \]

Therefore we have only to worry about the contributions from \( x = 0 \). This gives

\[ \hat{f}(\omega) = \frac{a}{\sqrt{2\pi}} \left( \frac{1}{a-i\omega} - 0 \right) - \frac{1}{a+i\omega} \right) \]

\[ = \frac{a}{\sqrt{2\pi}} \left( \frac{1}{a-i\omega} + \frac{1}{a+i\omega} \right) \]

\[ = \frac{a}{\sqrt{2\pi}} \frac{2a}{a^2 + \omega^2} \]

\[ = \frac{2}{\sqrt{2\pi}} \frac{a^2}{a^2 + \omega^2} \]

Again the value of \( a \) just stretches out the \( \omega \) axis. \( \square \)
3. What do the functions from the last problem look like for large positive $a$? What do the absolute values of their Fourier transforms look like? What about for small positive $a$? Might this have anything to do with Heisenberg uncertainty?

**Solution:** Since the parameter $a$ rescales in the $x$ axis while it rescales out the $\omega$ axis, we see that large $a$ gives a highly concentrated blip in the $x$ variable in each of the two problems above, while it spreads out the $\omega$ variable, and conversely small $a$ spreads out the blip $f(x)$ in the $x$ variable to a long squat function, while it concentrates the function $\hat{f}$ in the $\omega$ variable. Pictures of the square wave from 1 (a) are indicated in figures 51 on the following page and 53 on page 125 while the Fourier transforms appear in figures 52 on page 124 and 54 on page 126. Notice the direction of the parameter $a$. A picture of the Feyer function from 1 (b) is given in figure 55 on page 127 and its Fourier transform is in figure 56 on page 128 □

4. Write out the integral that you would use to calculate the Fourier transform $\hat{f}(\omega)$ of

$$f(x) = e^{-ax^2/2}.$$ 

Now use standard algebra of exponential functions to write it as the integral of the exponential of something. That something is quadratic in $x$. Complete the square to express it as the square of a linear function of $x$ plus a constant. Now use simple change of variable to solve the integral. You will need to know

$$\int_{-\infty}^{\infty} e^{-(x+c)^2/2} dx = \sqrt{2\pi}.$$

for any constant $c$ (real or complex). (This is easy to prove in complex variables class, next semester.)

**Solution:**

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2/2} e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a(x^2+2i\omega x)/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+i\omega/\sqrt{a}\right)^2/2-a\omega^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2a} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+i\omega/\sqrt{a}\right)^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2a} \int_{-\infty}^{\infty} e^{-\left(u+i\omega/\sqrt{a}\right)^2/2} \frac{du}{\sqrt{a}}$$
Figure 51: The square wave, with various values of $a$. Maple makes it look a bit triangular, but it should be square.
Figure 52: The Fourier transform of the square wave from the previous figure.
Figure 53: The square wave again, this time with the parameter $a$ decreasing.
Figure 54: The Fourier transform of the square wave from the previous figure. Notice that the parameter $a$ is decreasing here.
Figure 55: The Feyer function, getting sharper for larger $a$. 
Figure 56: The Fourier transform of the Feyer function, getting broader and flatter for larger $a$. 
where
\[ u = \sqrt{ax} \]
\[ du = \sqrt{a} dx \]
\[ dx = \frac{du}{\sqrt{a}} \]

So
\[
\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{e^{-\omega^2/2a}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-(u+i\omega/\sqrt{a})^2/2} du \\
= \frac{1}{\sqrt{2\pi}} \frac{e^{-\omega^2/2a}}{\sqrt{a}} \sqrt{2\pi} \\
= \frac{e^{-\omega^2/2a}}{\sqrt{a}}
\]

\[ \square \]

11.1 Fourier transforms and derivatives

Let’s find the Fourier transform of the derivative of a function:
\[
\mathcal{F}\left( \frac{df}{dx} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dx} e^{-i\omega x} dx.
\]

But now consider integration by parts: the idea is
\[
\frac{d}{dx} (f(x)e^{-i\omega x}) = \frac{df}{dx} e^{-i\omega x} + f(x)(-i\omega)e^{-i\omega x}.
\]

So we can rewrite our integral as
\[
\mathcal{F}\left( \frac{df}{dx} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{df}{dx} e^{-i\omega x} \right) dx - f(x)(-i\omega)e^{-i\omega x} \int_{-\infty}^{\infty} e^{-i\omega x} dx.
\]

This splits into two integrals:
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dx} (f(x)e^{-i\omega x}) dx + i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx.
\]

The second one is just \( i\omega \mathcal{F}(f) \). The first one is
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dx} (f(x)e^{-i\omega x}) dx = \frac{1}{\sqrt{2\pi}} \left. f(x)e^{-i\omega x} \right|_{x=-\infty}^{x=\infty},
\]

Lets suppose that the function \( f(x) \) dies off for large \( x \). This is physically reasonable for most problems. Then this stuff just disappears, and we get
\[
\mathcal{F}\left( \frac{df}{dx} \right) = i\omega \mathcal{F}(f).
\]

This is the key to lots of PDE problems: under Fourier transform, the derivative operation turns into the operation of multiplication by \( i\omega \). So it turns calculus into algebra.
11.2 Convolution

Fourier transforms take sums to sums, and derivatives to multiplication by $i\omega$. What corresponds to multiplication under the Fourier transform? Let

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) \, dy$$

called the convolution of $f$ with $g$. We want to show that

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

so convolution becomes multiplication. The proof is easy:

$$\mathcal{F}(f * g)(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f * g(x)e^{-ipx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) \, dy \right) e^{-ipx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)e^{-ipx} \, dx \right) g(y) \, dy$$

(changing the order of integration)

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{-ip(z+y)} \, dz \right) g(y) \, dy$$

(changing variable of integration: $z = x - y$)

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{-ipz} \, dz \right) e^{-ipy} g(y) \, dy$$

$$= \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)e^{-ipz} \, dz \right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)e^{-ipy} \, dy$$

$$= \mathcal{F}(f)\mathcal{F}(g).$$

11.2.1 What is convolution?

In figure [57] on the following page, we have a wall, and we are listening to sounds coming from the other side of it. Let $f(t)$ be the sound emitted. Then what we hear is muffled, and turns out to be (roughly) a convolution $f * h(t)$ for some function $h$ which depends on material properties of the wall. Why would this be? As sound enters the wall, it bounces back and forth inside the wall, so part of it will come almost right away, but some of it may stay inside the wall for a while, bouncing back and forth, before it gets out. So what you hear at time $t$ is not $f(t)$ but a mix of a little of $f(t)$ with a little of $f(t-s)$ for various positive
Figure 57: Listening to sounds through a wall
values of $s$ (here $s$ represents how long the sound stayed in the wall). So you hear

$$f * h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t - s)h(s) \, ds$$

where $h(s)$ is how strongly you hear the sound from $s$ seconds ago (or what time units we are working in). So the function $h(s)$ in this case must look something like in figure 58. The sharp peak of $h(0)$ is due to most of the sound getting through right away; the function $h(s)$ is zero for negative $s$ because the sound can’t get through before it is generated. This general concept, that convolution represents smearing, is very fundamental.

Now suppose that I want to remove the muffling of noise from a signal $f(t)$. First, I get you to play a piano on the other side of the wall. When you play a pure note, say

$$f_{\text{sample}}(t) = e^{i\omega_0 t}$$

then I listen on the other side and hear

$$f_{\text{sample}} * h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_{\text{sample}}(t - s)h(s) \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega_0(t-s)}h(s) \, ds$$

$$= e^{i\omega_0 t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega_0s}h(s) \, ds$$

so that I can “hear” $\hat{h}(\omega_0)$. In fact at the instant $t = 0$ when you press the piano key, I hear $\hat{h}(\omega_0)$ precisely. Once I hear it for lots of different notes, I effectively
know $\hat{h}$. Then I can listen to any sound $f(t)$ coming through the wall. What I hear is $f * h(t)$. I record this, get a computer to Fourier transform it, and now I have $\mathcal{F}(f * h) = \tilde{f} \hat{h}$ in my computer. But I have already “heard” $\hat{h}$. So now I just divide $\tilde{f} \hat{h}/\hat{h} = \tilde{f}$, and I have $f$. I take the inverse Fourier transform to get $f(t)$. Now I have a recording of the original sound, free of noise.

Why doesn’t this work? For example, take the Earth to be our wall, and ask why we can’t hear what people are saying in China. The reason is that $h(s) = 0$, i.e. there is no sound coming through at all, so $\hat{h}(\omega) = 0$ too, and therefore when we divide $\tilde{f} \hat{h}/\hat{h}$ we are dividing zero by zero.

Similarly, if $\hat{h}(\omega)$ is very small for some frequency $\omega$, this means that very little of what goes on at that frequency gets through the wall. But it also says that we will have a lot of trouble dividing: remember that there is no accurate way to divide very small numbers, because of round off errors. So effectively, we can only recover the frequencies of $f(t)$ that get through the wall pretty strongly, i.e. with $\hat{h}(\omega)$ not too small.

### 11.3 Heat in infinite space

For now, we take just one space variable $x$. Let $u(x, t)$ be temperature at time $t$ position $x$. The heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Let $\hat{u}(p, t)$ be the Fourier transform in the $x$ variable, leaving the $t$ variable alone.

$$\hat{u}(p, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-ipx} \, dx.$$  

(We use $p$ instead of $\omega$ here, to avoid confusion, since we usually think of $\omega$ as a frequency of a process over time, not a spatial frequency.)

The $t$ variable is left alone, and since the Fourier transform is a linear transform, it gives

$$\frac{\partial \hat{u}}{\partial t} = \frac{\partial \hat{u}}{\partial t}.$$

But we transformed the space variable $x$, so

$$\frac{\partial \hat{u}}{\partial x} = ip \hat{u}$$

and taking two derivatives gives

$$\frac{\partial^2 \hat{u}}{\partial x^2} = (ip)^2 \hat{u} = -p^2 \hat{u}.$$

So the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is

133
becomes

$$\frac{\partial \hat{u}}{\partial t} = -c^2 p^2 \hat{u}.$$  

We have derivatives now only in $t$. So for any fixed $p$ value this is an ordinary differential equation in $t$: an exponential decay. The solution is

$$\hat{u}(p, t) = e^{-c^2 p^2 t} \hat{u}(p, 0).$$

Again it is an exponential decay, at a very rapid rate in the high frequencies. How do we get the hats off? Use the convolution: Fourier transform takes convolution to multiplication, so inverse Fourier transform takes multiplication back to convolution:

$$u(x, t) = G(x, t) * u(x, 0)$$

where the convolution is taken in the $x$ variable only, and

$$G(x, t) = \mathcal{F}^{-1} \left( e^{-c^2 p^2 t} \right)$$

so we can write

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(p, t) e^{ipx} \, dp$$

and we have a nice expression:

$$u(x, t) = G(x, t) * u(x, 0) = \int u(s, 0) G(x - s, t) ds.$$

We still have to write $G$ in a nicer way: calculate the integral. Call $G$ the heat kernel.

### 11.3.1 Calculating the heat kernel

The heat kernel is an integral

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-c^2 p^2 t} e^{ipx} \, dp.$$

We can write it as

$$G(x, t) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \hat{G}(p, t) e^{ipx} \, dp$$

an inverse Fourier transform of the function

$$\hat{G}(p, t) = \frac{1}{\sqrt{2}} e^{-c^2 p^2 t}.$$

We have already calculated the Fourier transform of this function in the solved problems above, and the same approach gives the inverse transform:

$$G(x, t) = \frac{1}{2c \sqrt{\pi t}} e^{-x^2 / 4c^2 t}.$$
So the general solution of the heat equation in the infinite line is

\[ u(x, t) = \int_{-\infty}^{\infty} f(s)G(x - s, t) \, ds \]

where

\[ G(x, t) = \frac{1}{2c\sqrt{\pi t}} e^{-x^2/4c^2t} \]

and \( f(s) \) is the temperature at time \( t = 0 \).

Notice: we don’t calculate any amplitudes—just write down one integral, and get the answer.

11.4 Wave equation
d’Alembert’s solution still works, which is pretty obvious since it works for periodic systems of any period. Solution: if \( u(x, t) \) satisfies the wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

with initial position \( u(x, 0) = f(x) \) and initial velocity \( \frac{\partial u}{\partial t}(x, 0) = g(x) \) then

\[ u(x, t) = \frac{1}{2} f(x - ct) + \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \]

The period never appeared in the formula anyway.

11.5 The heat kernel in one and two dimensions
We showed using the Fourier transform that if \( u(x, t) \) is the temperature at time \( t \) of an infinite length (or very long) wire parameterized by \( x \) then

\[ u(x, t) = \int_{-\infty}^{\infty} u(s, 0)G(t, x - s) \, ds \]

where

\[ G(t, x) = \frac{1}{2c\sqrt{\pi t}} \exp \left( -\frac{x^2}{4c^2t} \right). \]

The kernel \( G(t, x) \) is the solution of the heat equation when the initial temperature is concentrated at a single point, with one unit of total heat. See figure 69 on the following page. The expression for \( u(x, t) \) above says that any solution of the heat equation is given by treating the initial temperature \( u|_{t=0} \) as a sum of heat impulses at each point \( x \), so that the solution is a sum of kernels \( G(t, x - s) \) multiplied by the values \( u(s, 0) \), the total heat of each of those heat impulses.

Solved Problems #9

135
Figure 59: The heat kernel in one space dimension. Note that the spike at $t = 0$ is actually infinitely high.
1. If \( u(x, y, t) \) is the temperature at time \( t \) of an infinite plane parameterized by \( x, y \), show that

\[
u(x, y, t) = \int \int u(X, Y, 0) G(t, x - X, y - Y) dX dY
\]

where the integral is carried out over the entire plane, and

\[
G(t, x, y) = G(t, x) G(t, y) = \frac{1}{4c^2 \pi t} \exp \left( -\frac{x^2 + y^2}{4c^2 t} \right)
\]

Solution: The steps are identical to those in the one variable problem, and it all comes down to \( e^{A+B} = e^A e^B \).

11.6 The electrostatic kernel in a half plane

Along the bottom edge of a very large rectangular metal plate, we put an electric source. What is the potential inside the plate? Suppose that the plate is so large that we can pretend it is the upper half plane in the \( x, y \) plane, i.e. the points with \( y \geq 0 \). We fix the values of the potential \( u(x, y) \) at \( y = 0 \), on the bottom. Electrostatic potentials satisfy the Laplace equation:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
\]

Starting with the data \( u(x, 0) \) from the bottom, we use this PDE to push up into the inside of the plate.

Let \( \hat{u}(p, y) \) be the Fourier transform just in the \( x \) variable, leaving \( y \) alone:

\[
\hat{u}(p, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ipx} dx.
\]

Then \( \frac{\partial}{\partial x} \) becomes \( ip \) and so the Laplace equation says

\[
0 = (ip)^2 \hat{u} + \frac{\partial^2 \hat{u}}{\partial y^2}.
\]

(It is the same \( y \) variable.) Rewrite this as

\[
\frac{\partial^2 \hat{u}}{\partial y^2} - p^2 \hat{u} = 0.
\]

This is an ordinary differential equation in \( \hat{u} \), for each fixed value of \( p \): it is a harmonic oscillator, with negative spring constant, and so the solution is

\[
\hat{u}(p, y) = Ae^{-py} + Be^{py}.
\]
Figure 60: Place a charge on the bottom edge of a very large plate: what happens inside?
These $A$ and $B$ are constants. But that is for each fixed $p$. If we let $p$ vary, then $A$ and $B$ can vary with $p$:

$$\hat{u}(p, y) = A(p)e^{-py} + B(p)e^{py}.$$ 

We will assume for good physics that the potential $u$ dies off far away from the sources, so for large $y$, $u \to 0$. But the $e^{py}$ term grows rapidly with $y$, when $p > 0$, so we need $B(p) = 0$ when $p > 0$. Similarly, when $p < 0$ the $e^{-py}$ grows rapidly, so we need $A(p) = 0$ when $p < 0$. For each $p$ we either get an $A$ contribution if $p < 0$ or a $B$ contribution if $p > 0$. So we can write

$$\hat{u}(p, y) = C(p)e^{-|p|y}$$

where

$$C(p) = \begin{cases} A(p) & \text{if } p > 0 \\ B(p) & \text{if } p < 0. \end{cases}$$

Plug in $y = 0$ (looking at the bottom of the plate):

$$\hat{u}(p, 0) = C(p).$$

So $C(p)$ is just the Fourier transform of $u(x, 0)$, the potential on the bottom edge:

$$\hat{u}(p, y) = \hat{u}(p, 0)e^{-|p|y}.$$ 

Now inverse Fourier transform and get a convolution:

$$u(x, y) = u(x, 0) * F^{-1}\left(e^{-|p|y}\right)(x).$$

We know the inverse Fourier transform of $e^{-|p|y}$ is

$$F^{-1}\left(e^{-|p|y}\right)(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.$$ 

If we have a wire carrying charge along the $x$ axis, then to find the electrostatic potential in the half plane where $y > 0$ we can use the integral

$$u(x, y) = \int_{-\infty}^{\infty} u(X, 0) P(x - X, y) \, dX$$

where

$$P(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}. $$

The kernel $P(x, y)$ is the potential generated in the half plane by a unit charge placed at the origin. See figure 61 on the next page. If we shift around the location of this charge, and scale it by an amount $u(x, 0)$, then we get a contribution of this amount multiplied by $P$ in our integral for $u(x, y)$.

**Solved Problems #10**
Figure 61: The Poisson kernel in a half plane. Note that the spike at $y = 0$ is actually infinitely high.
Figure 62: A pipe with the $x$ and $y$ axes labelled. A wire carries charge along the $x$ axis.

1. Show that
\[
\frac{1 - e^{-x}}{1 - e^x} = -e^{-x}
\]

2. A infinitely long wire is laid along an infinitely long radius $r$ pipe. See figure 62. Use the Fourier transform to find the solution $u(x, y)$ of the Laplace equation
\[
0 = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
\]
in the form
\[
u(x, y) = \int_{-\infty}^{\infty} u(s, 0) P_r(x - s, y) \, ds
\]
where we ask that the $y$ coordinate be periodic with period $2\pi r$ (where $r$ is the radius of the pipe). Hints:

- Take Fourier transform in $x$.
- The Laplace equation in $u$ becomes an ODE in $\hat{u}$; solve it. You end up with two arbitrary amplitude functions of $\omega$.
- use the fact that $u(x, 2\pi r) = u(x, 0)$ to determine relations on the amplitudes of $\hat{u}$
- Use the result of problem 1. to solve for one amplitude function in terms of the other.
- Since the $y$ variable is now wrapped around the pipe, $y$ only ranges from 0 to $2\pi r$, so now there is no problem with exponential growth or decay. Don’t worry about that.
- If
\[
\hat{f}(\omega) = \frac{e^{a\omega} + e^{-a\omega}}{e^{b\omega} + e^{-b\omega}}
\]
then
\[
f(x) = \sqrt{\frac{2\pi}{b}} \frac{\cos \left( \frac{\pi a}{2b} \right) \cosh \left( \frac{\pi x}{2b} \right)}{\cosh \left( \frac{\pi a}{b} \right) + \cos \left( \frac{\pi a}{b} \right)}
\]
(an inverse Fourier transform that even Maple doesn’t know).
• The kernel should turn out to be

\[ P_r(x, y) = \frac{1}{\pi r} \sin \left( \frac{y}{r} \right) \cosh \left( \frac{x}{r} \right) - \cos \left( \frac{y}{r} \right) \cosh \left( \frac{x}{r} \right) \]

and it looks like figure 63 on page 144.

**Solution:** Let

\[ \hat{u}(p, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ipx} \, dx. \]

The Laplace equation becomes

\[ \frac{\partial^2 \hat{u}}{\partial y^2} - p^2 \hat{u} = 0 \]

and the solutions to this are

\[ \hat{u} = A(p)e^{-py} + B(p)e^{py} \]

just like for the half plane. We need

\[ u(x, 0) = u(x, 2\pi r) \]

so that

\[ \hat{u}(p, 0) = \hat{u}(p, 2\pi r). \]

This turns into

\[ A(p) + B(p) = A(p)e^{-2\pi r \omega} + B(p)e^{2\pi r \omega} \]

so that

\[ B(p) = -A(p) \frac{1 - e^{-2\pi r p}}{1 - e^{2\pi r p}} \]

and using problem 1, this is

\[ B(p) = A(p)e^{-2\pi r p}. \]

This gives

\[ \hat{u}(p, y) = A(p) \left( e^{-py} + e^{py - 2\pi r p} \right). \]

At \( y = 0 \) we have

\[ \hat{u}_{|y=0} = A(p) \left( 1 + e^{-2\pi r p} \right) \]

so that at any \( y \) value

\[ \hat{u}(p, y) = \hat{u}(p, 0) \frac{e^{-py} + e^{(y - 2\pi r)p}}{1 + e^{-2\pi r p}} \]
Multiply numerator and denominator by $e^{\pi rp}$:

$$\hat{u}(p, y) = \hat{u}(p, 0) \frac{e^{(\pi r-y)p} + e^{-(\pi r-y)p}}{e^{\pi rp} + e^{-\pi rp}}.$$ 

So taking inverse Fourier transform,

$$u(x, y) = u(x, 0) * \mathcal{F}^{-1} \left( \frac{e^{(\pi r-y)p} + e^{-(\pi r-y)p}}{e^{\pi rp} + e^{-\pi rp}} \right)$$

$$= \int_{-\infty}^{\infty} u(s, 0) P_r(x-s, y) \, ds$$

where

$$P_r(x, y) = \mathcal{F}^{-1} \left( \frac{e^{(\pi r-y)p} + e^{-(\pi r-y)p}}{e^{\pi rp} + e^{-\pi rp}} \right)$$

(inverse Fourier transform) but we know how to calculate such a thing from the hints: plug in

$$a = \pi r - y$$

$$b = \pi r$$

and you get

$$P_r(x, y) = \frac{1}{\pi r} \frac{\cos \left( \frac{\pi}{2} - \frac{y}{2r} \right) \cosh \left( \frac{x}{2r} \right)}{\cosh \left( \frac{\pi}{2r} \right) + \cos \left( \pi - \frac{y}{r} \right)}$$

and with a little trigonometry, this becomes the required solution. □

3. What happens in the limit as $r \to \infty$? Hint: you need to know the Taylor expansions

$$\sin x \sim x$$

$$\cos x \sim 1 - \frac{x^2}{2}$$

$$\cosh x \sim 1 + \frac{x^2}{2}$$

and you should recover the kernel for the half plane problem we found in class.

Solution: We easily find that as $r \to \infty$,

$$P_r(x, y) \to \sqrt{\frac{2}{\pi}} \frac{y}{x^2 + y^2} = P(x, y)$$

becomes the kernel for the half plane, by plugging in the Taylor expansions and multiplying numerator and denominator by $r^2$. □
Figure 63: The kernel for the electrostatic potential of a wire laid along a pipe. The pipe has unit radius $r = 1$. Note that the spike at $y = 0$ and the one at $y = 2\pi r = 2\pi$ are both actually infinitely high.
Figure 64: A function defined only on a half line

Figure 65: Even extension of figure [64]
11.7 Half lines

What can we do with functions $f(t)$ like figure 64 on the preceding page? Try extending it to an even function, as in figure 65 on the page before. Then we can write it as

$$f(t) = \int_{0}^{\infty} (A(\omega) \cos(\omega t) + B(\omega) \sin(\omega t)) \, d\omega$$

and as usual $B(\omega) = 0$ because the function is even. But then

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos(\omega t) \, dt$$

$$= \frac{2}{\pi} \int_{0}^{\infty} f(t) \cos(\omega t) \, dt.$$  

To make the constants more symmetrical, we will define

$$\hat{f}_c(\omega) = \sqrt{\frac{\pi}{2}} A(\omega)$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos(\omega t) \, dt$$

called the Fourier cosine transform. The factor on the front doesn’t matter much, but it gives

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_c(\omega) \cos(\omega t) \, d\omega.$$

The only reason for rescaling the constant on the front was to get these two formulas to look exactly the same. We will also write $\hat{f}_c$ as $\mathcal{F}_c[f]$.

If we extend to an odd function, we get figure 66 on the next page. This gives rise to the Fourier sine transform:

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin(\omega t) \, dt$$

so that

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{f}_s(\omega) \sin(\omega t) \, d\omega.$$

We will also write $\hat{f}_s$ as $\mathcal{F}_s[f]$.

Example: Let $f(t) = e^{-t}$. Then

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-t} \cos(\omega t) \, dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1 + \omega^2}. $$
11.7.1 Derivatives

What is the cosine transform of the derivative of a function?

\[
\mathcal{F}_c \left[ \frac{df}{dt} \right] (\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{df}{dt} \cos(\omega t) \, dt = \sqrt{\frac{2}{\pi}} f(t) \cos(\omega t) \bigg|_{t=0}^{t=\infty} \left. - \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) (-\omega \sin(\omega t)) \, dt \right]
\]

by integration by parts. But \( f(t) \) dies off as \( t \to \infty \), we always assume, so at \( t = \infty \) we get \( f(t) = 0 \), and

\[
\mathcal{F}_c \left[ \frac{df}{dt} \right] (\omega) = -\sqrt{\frac{2}{\pi}} f(0) - \omega \mathcal{F}_s [f](\omega).
\]

Similarly

\[
\mathcal{F}_s \left[ \frac{df}{dt} \right] = -\omega \mathcal{F}_c [f](\omega).
\]

Now we can apply these formulae twice to take second derivatives:

\[
\mathcal{F}_c \left[ \frac{d^2f}{dt^2} \right] = -\sqrt{\frac{2}{\pi}} \frac{df}{dt} (0) - \omega^2 \mathcal{F}_c [f](\omega),
\]

\[
\mathcal{F}_s \left[ \frac{d^2f}{dt^2} \right] = \sqrt{\frac{2}{\pi}} \omega f(0) - \omega^2 \mathcal{F}_s [f](\omega).
\]
11.7.2 Heat in a half infinite wire

Take a half line, the positive $x$ axis, think of it as a wire, and heat it. Keep the $x = 0$ left end at $0^\circ$. The temperature $u$ satisfies

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$ 

Let

$$\hat{u}_s(p, t) = \text{Fourier sine transform in the } x \text{ variable}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, t) \sin(px) \, dx.$$ 

Then the derivatives in $t$ transform to derivatives in $t$, while derivatives in $x$ transform according to formulae above:

$$\mathcal{F}_s \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \hat{u}_s$$

$$\mathcal{F}_s \frac{\partial^2 u}{\partial x^2} = \sqrt{\frac{2}{\pi}} p u(0, t) - p^2 \hat{u}_s(p, t).$$

But $u(0, t) = 0$ because we keep the left end at $0^\circ$. So the heat equation becomes, after Fourier sine transform,

$$\frac{\partial \hat{u}_s}{\partial t} = -c^2 p^2 \hat{u}_s(p, t).$$

This is just an ordinary differential equation in $t$, and the solution is

$$\hat{u}_s(p, t) = e^{-tc^2p^2} \hat{u}_s(p, 0).$$

To recover $u$ from $\hat{u}_s$ we use

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_s(p, t) \sin(px) \, dp$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-tc^2p^2} \hat{u}_s(p, 0) \sin(px) \, dp$$

and now plug in the formula to find $\hat{u}_s(p, 0)$ from $u(x, 0)$:

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-tc^2p^2} \left( \sqrt{\frac{2}{\pi}} u(X, 0) \sin(pX) \, dX \right) \sin(px) \, dp$$

and now change the order of integration:

$$= \int_0^\infty u(X, 0) G(x, X, t) \, dX$$

148
where

\[ G(x, X, t) = \frac{2}{\pi} \int_0^\infty e^{-tc^2 p^2} \sin(px) \sin(pX) \, dp. \]

This integral is not too difficult—it gives

\[ G(x, X, t) = \frac{1}{c\sqrt{\pi t}} \left( \exp \left( -\frac{1}{4tc^2} (x - X)^2 \right) \right) - \exp \left( -\frac{1}{4tc^2} (x + X)^2 \right) \]

...So to find \( u(x, t) \) just plug \( u(x, 0) \) into this integral with \( G(x, X, t) \).