6.1 The Natural Logarithm Function

\[ D_x \left( \frac{x^3}{3} \right) = x^2 \]
\[ D_x \left( \frac{x^2}{2} \right) = x \]
\[ D_x (x) = 1 \]
\[ D_x (\frac{1}{x}) = \frac{1}{x^2} \]
\[ D_x \left( \frac{1}{2x^2} \right) = \frac{1}{x^3} \]

---

**Defn** \( \ln x = \int_1^x \frac{1}{t} \, dt \), \( x > 0 \)

Remember graph of \( y = \ln x \)

---

Area (accumulation)

---

From 1st Fundamental Thm of Calculus

\[ D_x \left( \int_1^x \frac{1}{t} \, dt \right) = D_x (\ln x) = \frac{1}{x} \], \( x > 0 \)

---

Ex 1 Find \( \frac{dy}{dx} \) if \( y = \ln(x^2) \)
6.1 (cont)

Ex 2: Find \( \frac{dy}{dx} \) and state domain.

(a) \( y = \ln (3\sqrt{2x}) \)

(b) \( y = \ln (3x^2 + 14x - 5) \)

\[ D_x [\ln |x|] = \frac{1}{x}, \quad x \neq 0 \]

Proof:  
1. If \( x > 0 \) then \( |x| = x \). 
   \( \Rightarrow D_x [\ln |x|] = D_x [\ln x] = \frac{1}{x} \) (by previous box pg. 10)

2. If \( x < 0 \) then \( |x| = -x \).
   \( \Rightarrow D_x (\ln |x|) = D_x [\ln (-x)] = (\frac{1}{-x})(-1) = \frac{1}{x} \)
Ex. 3 Evaluate integrals.

(a) \( \int \frac{6}{3x-2} \, dx \)

(b) \( \int_2^5 \frac{3x}{7-2x^2} \, dx \) (Note: this integral is valid because \( 7-2x^2 > 0 \) \( \forall x \in [2, 5] \).)
Properties of Logarithms

1. \( \ln 1 = 0 \)
2. \( \ln(ab) = \ln a + \ln b \)
3. \( \ln \left( \frac{a}{b} \right) = \ln a - \ln b \)
4. \( \ln a^r = r \ln a \)

Proof
1. \( \ln 1 = \int_1^1 \frac{1}{t} \, dt = 0 \) (area under curve)
2. \( D_x (\ln(ax)) = \frac{1}{ax} (a) = \frac{1}{x} \) and \( D_x (\ln x) = \frac{1}{x} \)
   
   \( \Rightarrow D_x (\ln(ax)) = D_x (\ln x) \)
   
   \( \Rightarrow \ln(ax) = \ln x + C \quad \forall x > 0 \)

   if \( x = 1 \), then \( \ln a = \ln 1 + C \)
   
   \( \Rightarrow C = \ln a \)

   \( \Rightarrow \ln(ax) = \ln x + \ln a \)

   We can prove 3 + 4 similarly.

Ex 4 (#30) Rewrite as single log.

\[ \ln (x^2 - 9) - 2 \ln (x-3) - \ln (x+3) \]
6.1 (cont)

Ex 5 (#34) Find \( \frac{dy}{dx} \) by logarithmic differentiation.

\[
y = \frac{(x^2+3)^{2/3} \cdot (3x+2)^2}{\sqrt{x+1}}
\]

\[
\Rightarrow \ln y = \ln \left[ \frac{(x^2+3)^{2/3} \cdot (3x+2)^2}{\sqrt{x+1}} \right]
\]
6.2 Inverse Fns & Their Derivatives

If \( f(x) \) is a function, then \( f^{-1}(x) \) is notation for the inverse of \( f \).
(read as "\( f \) inverse"; \( f^{-1} \neq \frac{1}{f} \))

If we have complete graph of \( f(x) \), we can use horizontal line test to check for existence of \( f^{-1}(x) \).

\[ f(x) = x^3 \implies f^{-1}(x) = \sqrt[3]{x} \quad \text{(i.e. way to "undo" cubing is to take the cube root)} \]

If \( f(x) \) \& \( f^{-1}(x) \) are inverses, then \( (f \circ f^{-1})(x) = x \)
\[ = (f^{-1} \circ f)(x) \]

- Inverses exist when we can get back to an \( x \) given a \( y \), i.e. \( x = f^{-1}(y) \iff f(x) = y \).

- Inverses exist when a fn is \underline{one-to-one} (which means \( x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \))

If we don't have graph, how can we algebraically test if a fn has an inverse?

Then A: If \( f \) is strictly monotonic on its domain, then \( f \) has an inverse.

- Domain of \( f = \text{range of } f^{-1} \)
- Range of \( f = \text{domain of } f^{-1} \)
6.2 (cont)

Ex 1. Show that \( f(x) = 3x^7 + 4x^3 + x - 3 \) has an inverse.

(notice I didn't ask you to find the inverse.)

Ex 2. Explore whether or not \( f(x) = x^2 - 4 \) has an inverse. If not, can we restrict the domain so it does? If so, find \( f^{-1}(x) \).
Ex 3 Find $f^{-1}(x)$ for $y = \frac{2x-1}{3+5x}$, and check your work.

Graph of $f^{-1}(x)$ is $f(x)$ reflected across $y = x$ line!

You can see slope at $(d, c)$ is reciprocal of slope at $(c, d)$

$$\Rightarrow (f^{-1})'(d) = \frac{1}{f'(c)}$$
6.2 (cont.)

**Thm B: Inverse Function Thm**

If \( f \) is differentiable, strictly monotonic on \( I \), and \( f'(x) \neq 0 \) at some \( x \in I \), then \( f^{-1}(x) \) is differentiable at corresponding points \( y = f(x) \) in range of \( f \) and

\[
(f^{-1})'(y) = \frac{1}{f'(x)}.
\]

(i.e. \( \frac{dx}{dy} = \frac{1}{dy/dx} \))

**Ex 4** For \( f(x) = x^5 + 5x - 4 \), find \( (f^{-1})'(2) \) using Thm B.
6.3 The Natural Exponential Function

Remember graph of \( y = \ln x \).

\( y = \ln x \)  
\[ \Rightarrow \]  
\( \text{it is strictly monotonic} \)  
\[ \Rightarrow \]  
\( \text{it has an inverse fn} \).  
\[ \Rightarrow \]  
\( \text{Can we draw the universe fn's graph?} \)

We'll name the universe fn "exp."

\[ \Rightarrow x = \exp(y) \Leftrightarrow y = \ln x \]

\[ \text{domain of } \ln x : x > 0 \]
\[ = \text{range of } \exp(x) : y > 0 \]
\[ \text{domain of } \exp(x) : x \in \mathbb{R} \]
\[ = \text{range of } \ln x : y \in \mathbb{R} \]

Since "\( \ln \)" and "\( \exp \)" are inverse fn's,

\[ \ln(\exp(x)) = \exp(\ln(x)) = x. \]

**Defn.** Let \( e \in \mathbb{R} \) denote unique \# \( e \) s.t. \( \ln e = 1 \).

\( e \) is irrational; named after Euler; \( e \approx 2.718; \)
we know \( e \) exists because we can see it on graph above.

Important understanding \( \Rightarrow \)

\[ r \in \mathbb{R} \]
\[ \exp(r) = \exp(r \ln e) \]
\[ = \exp(\ln e^r) \]
\[ = e^r \]

(since \( \exp \) and "\( \ln \)" are inverses)

very cool!
6.3 (cont)

**Proposition**
Let $a, b \in \mathbb{R}$. Then $e^{a+b} = e^a \cdot e^b$.

**Proof**
\[
\frac{e^a}{e^b} = \exp \left( \ln \left( \frac{e^a}{e^b} \right) \right) \\
= \exp \left( \ln e^a - \ln e^b \right) \\
= \exp \left( a \ln e - b \ln e \right) \\
= \exp (a-b) = e^{a-b}
\]

Let $y = e^x$ ($\Rightarrow \ln y = x$).

\[
\Rightarrow \frac{d}{dx} (\ln y) = \frac{d}{dx} (x) \\
\frac{1}{y} \frac{dy}{dx} = 1 \\
y \frac{dy}{dx} = y = e^x
\]

\[
\Rightarrow D_x (e^x) = e^x \\
\text{or} \\
D_x (e^{ux}) = e^{ux} \cdot u \\
\text{Chain Rule}
\]

**Ex 1** Find $y'$. $y = e^{x^2 - 3x}$
6.3 (cont)

Ex 2 Find $y'$, \( y = e^{\sqrt{x} \ln x} \)

Ex 3 For \( f(x) = e^x - e^{-x} \), analyze the graph.
(i.e. min, max pts, concavity, pts of inflection, sketch graph)
6.3 (cont)

Since $D_x[e^x] = e^x$, then $\int e^x \, dx = e^x + C$

Ex 4 Evaluate integrals.

(a) $\int e^{bx} \, dx$

(b) $\int e^{x} + e^{x} \, dx$

(c) $\int_{1}^{2} \frac{e^{3/x}}{x^2} \, dx$
6.4 General Exponential and logarithmic Functions

\[
\begin{align*}
\text{at } R \quad a^x &= \exp (\ln a^x) = \exp (x \ln a) = e^{x \ln a} \\
&= \left\{ \begin{array}{l l}
\quad & x \ln a \quad \text{for } a > 0 \\
\quad & x \in \mathbb{R}
\end{array} \right. \\
\Rightarrow \ln (a^x) &= \ln (e^{x \ln a}) = x \ln a \quad \text{(proof of log property)}
\end{align*}
\]

Properties of Exponents

(i) \( a^{x+y} = a^x a^y \)

(ii) \( a^{x-y} = \frac{a^x}{a^y} \)

(iii) \( (a^x)^y = a^{xy} \)

(iv) \( (ab)^x = a^x b^x \)

(v) \( \left( \frac{a}{b} \right)^x = \frac{a^x}{b^x} \)

\begin{align*}
Pf \ (v) \quad \left( \frac{a}{b} \right)^x &= e^{\ln \left( \frac{a}{b} \right)^x} \\
&= e^{x \ln \left( \frac{a}{b} \right)} \\
&= e^{x \left( \ln a - \ln b \right)} \\
&= e^{x \ln a - x \ln b} \\
&= e^{x \ln a - x \ln b} \\
&= e^{x \ln a - x \ln b} \\
&= e^{x \ln a - x \ln b} \\
&= e^{x \ln \frac{a}{b}} \\
&= \frac{a^x}{b^x} \quad \text{//}
\end{align*}

\[
\begin{align*}
D_x [a^x] &= D_x \left[ e^{x \ln a} \right] = e^{x \ln a} (\ln a) = a^x (\ln a) \\
\text{and } \int a^x \, dx &= \int e^{x \ln a} \, dx \quad \text{let } u = x \ln a \\
&= \frac{1}{\ln a} e^u + C \\
&= \frac{1}{\ln a} e^{x \ln a} + C = \frac{1}{\ln a} a^x + C
\end{align*}
\]
6.4 (cont.)

\[
\begin{align*}
\text{Ex 1} & \quad \text{Find } y'. \\
& \quad y = (2x^3 + 9x)^4 + 4^{2x^3 + 9x}
\end{align*}
\]

\[
\begin{align*}
\text{Ex 2} & \quad \text{Evaluate } \int \frac{2\sqrt{x}}{3\sqrt{x}} \, dx
\end{align*}
\]
6.4 (cont.)

Remember log defn (from algebra)

\[ y = \log_a x \iff a^y = x \]

\[ \implies \ln(a^y) = \ln x \]

\[ \implies y \ln a = \ln x \]

\[ \implies y = \frac{\ln x}{\ln a} = \log_a x \]

charge of base formula

\[ \implies D_x(\log_a x) = D_x \left( \frac{\ln x}{\ln a} \right) = \left( \frac{1}{\ln a} \right) \left( \frac{1}{x} \right) \]

\[ \text{Very cool} \implies \text{We know } D_x(x^a) = ax^{a-1} \text{ if } a \in \mathbb{Q} \]

but what if \( a \) is irrational?

\[ D_x(x^a) = D_x(e^{a \ln x}) = e^{a \ln x} \left( \frac{a}{x} \right) = x^a \left( \frac{a}{x} \right) = ax^{a-1} \]

\[ D_x(x^a) = ax^{a-1} \quad \forall a \in \mathbb{R} \]

Ex 3 Find \( y' \): \[ y = \sin^2 x + 2^{\sin x} \]
Ex 4. Find $y'$.

$y = (\ln x^2)^{2x+3}$ (Hint: We must take ln on both sides first.)

Ex 5. Evaluate $\int_0^1 \left(10^{3x} + 10^{-3x}\right) \, dx$. 

[Handwritten Math Note: 17]

(Math 1220)
6.4 (cont)

Ex 6 If \( y = x^x \), find \( y' \). (Tricky)
6.5 Exponential Growth & Decay

Population Growth ⇒ let \( y_0 = \) population at start
then \( \frac{dy}{dt} = ky \) is reasonable assumption
(i.e. the rate of growth, or decay, is proportional to the population)

⇒ \( dy = ky \, dt \)
\[ \int y \, dy = \int k \, dt \]
\[ \ln |y| = kt + c \]
\[ \ln y = kt + c \]
\[ y = e^{kt+c} \]
\[ y = e^c e^{kt} \]
\[ y = y_0 e^{kt} \]
\[ \text{ } e^c \in \mathbb{R} \text{ (i.e. a constant)} \]
when \( t = 0, \ y = e^c \)
\( k > 0 \) ⇒ growth \[ + \]
\( k < 0 \) ⇒ decay \[ - \]

so we write

(There are other factors that affect population, like resources & land, so population more likely follows logistic model.

\[ \frac{dy}{dt} = ky \, (L-y) \]
\( L = \) limiting population

but for small \( y \), \( \frac{dy}{dt} \approx ky \))
Ex 1  Population of U.S. was 3.9 million in 1790 + 178 million in 1960. If the rate of growth is assumed proportional to the population, what estimate would you give for the population in 2000? Compare your answer with actual population of 275 million.
Ex 2. If a radio-active substance loses 15% of its radioactivity in 2 days, what is its half-life?

\[
\frac{0.08}{4} = 0.02 \text{ quarterly interest}
\]

Compound Interest

Put $100 in a bank with 8% interest compounded quarterly.

\[
\begin{array}{c|c}
\text{after qtr} & \text{value} \\
\hline
0 & 100 \\
1 & 100 \times (1 + 0.02) \\
2 & [100 \times (1 + 0.02)] \times (1 + 0.02) = 100 \times (1 + 0.02)^2 \\
3 & [100 \times (1 + 0.02)] \times (1 + 0.02 \times (1 + 0.02)) = 100 \times (1 + 0.02)^3 \\
\vdots & \vdots \\
\hline
n & \text{?}
\end{array}
\]
\[
A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}
\]

What if we compound continuously?

\[
\Rightarrow A(t) = \lim_{n \to \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = ?
\]

Then

\[
\lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e
\]

Proof: If \( f(x) = \ln x \), then \( f'(x) = \frac{1}{x} \) and \( f'(1) = 1 \).

Use definition of derivative for \( f(x) \).

\[
1 = f'(1) = \lim_{h \to 0} \left[\frac{f(1+h) - f(1)}{h}\right] = \lim_{h \to 0} \left[\frac{\ln(1+h) - \ln(1)}{h}\right] = \lim_{h \to 0} \left[\frac{\ln(1+h)}{h}\right] = \lim_{h \to 0} \left[\frac{1}{h} \ln(1+h)\right] = \lim_{h \to 0} \left[\ln \left(\left(1+h\right)^{\frac{1}{h}}\right)\right]
\]

But

\[
\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e
\]

i.e.

\[
\lim_{h \to 0} \left(1 + h\right)^{\frac{1}{h}} = e
\]
6.5 (cont.)

Let's go back to a question. (continuous compounding)

\[ A(t) = \lim_{n \to \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} \]

\[ = A_0 \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{nt} = A_0 \lim_{n \to \infty} \left(1 + \frac{r}{n}\right)^{\frac{nt}{n}} \]

\[ = A_0 \left[ \lim_{n \to \infty}\left(1 + \frac{r}{n}\right)^{\frac{nt}{n}} \right] = A_0 \left[ \lim_{n \to \infty}\left(1 + h\right)^{\frac{t}{h}} \right] \quad \text{as } n \to \infty \quad \frac{r}{n} \to 0 \]

let \( h = \frac{r}{n} \)

\[ A(t) = A_0 e^{rt} \]

continuous compounding

EX 3: If Methuselah's parents had put $100 in the bank for him at birth and he left it there, what would Methuselah have had at his death (969 years later) if interest was 4% compounded annually?
6.6 First Order Linear Differential Equations

We looked at one kind of D.E. to solve in Calc 1, where we could separate the variables and solve.

Now, we'll look at a different kind of D.E. that requires a different strategy to solve.

First Order Linear D.E.

\[ \frac{dy}{dx} + P(x)y = Q(x) \]

- 1st order because 1st derivative
- Linear because all operations on y are linear

General solution = family of all solutions for D.E.

Particular solution = solution when you are given an extra condition

Strategy: multiply by integrating factor e

\[ e^{\int P(x)dx} \]

Let's try it.

\[ e^{\int P(x)dx} \frac{dy}{dx} + e^{\int P(x)dx} P(x)y = e^{\int P(x)dx} Q(x) \]

\[ \frac{d}{dx} \left[ ye^{\int P(x)dx} \right] = e^{\int P(x)dx} Q(x) \]

\[ \int d \left[ ye^{\int P(x)dx} \right] = \int e^{\int P(x)dx} Q(x) \, dx \]

\[ ye^{\int P(x)dx} = \int e^{\int P(x)dx} Q(x) \, dx \]

\[ y = e^{-\int P(x)dx} \int e^{\int P(x)dx} Q(x) \, dx \]
Ex 1 Solve \( \frac{dy}{dx} + \frac{5}{x} y = \frac{\cos(2x)}{x^5} \) for \( x > 0 \).
Ex 2  Solve \( y' = e^{2x} - 3y \) given \( y = 1 \) when \( x = 0 \)
Ex 3 A tank initially contains 200 gallons of brine, with 50 pounds of salt in solution. Brine containing 2 pounds of salt per gallon is entering the tank at the rate of 4 gallons per minute and the tank is flowing out at the same rate. If the mixture in the tank is kept uniform by constant stirring, find the amount of salt in the tank at the end of 40 minutes.

\[
\frac{dy}{dt} = \text{rate in} - \text{rate out}
\]
6.6 (cont)

**Kirchhoff's Law (for simple electrical circuit)**

\[
\frac{dI}{dt} + RI = E(t)
\]

- \(R\): resistance (in Ohms, \(\Omega\))
- \(L\): inductance (in Henrys)
- \(E\): voltage (in Volts)
- \(I\): current (in Amperes)

**Ex 4** Find \(I\) as a function of time for the circuit shown, assuming the switch is closed and \(I = 0\) at \(t = 0\).

\[E = 120\sin(377t)\]

\(\Rightarrow R = 0\) (since there is no resistor)
6.7 Approximations for Differential Eqns

For some diff. eqns, we cannot find explicit (analytic) solution, so we have to look for solution numerically (approximate soln).

**Slope Fields:** A graphical representation of slopes, for \( y' = f(x, y) \).

**Ex:** Plot slope field for \( y' = -y \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( y' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

Can you plot solution that satisfies \( y(0) = 2 \)?
Example 1: Given the slope field for a differential equation of form $y' = f(x, y)$, sketch the solution that satisfies $y(1) = 3$. Find $\lim_{x \to \infty} y(x)$ and approximate $y(2)$. 

![Slope Field Diagram]
Let's say \( y(x) \) is solution \( y' = f(x, y) \) at \( y(x_0) = y_0 \).

We know at \((x_0, y_0)\), slope is 
\[ m = f(x_0, y_0) \]

If we choose some small \( h \), then we would expect \( y \)-value on tangent line to give approximate value for actual \( y \)-value.

**Tangent line:** \( y - y_0 = f(x_0, y_0)(x - x_0) \)
\[ y = y_0 + f(x_0, y_0)(x - x_0) \]

At \( x_1 \): \( y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + hf(x_0, y_0) \)
(\text{approximate } y \text{-value})

To get approximate \( y_2 \) value, we basically repeat process.
\[ y_2 = y_1 + hf(x_1, y_1) \]

So, we incrementally get \( y \)-values to approximate actual solution.
Euler's method
To approximate \( y' = f(x, y) \) \( \Rightarrow y(x_0) = y_0 \), choose step size \( h \) (small) and repeat following steps, \( n = 1, 2, 3, \ldots \):
1. \( x_n = x_{n-1} + h \)
2. \( y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1}) \)

\textbf{Ex 2.} Use Euler's method with \( h = 0.2 \) to approximate \( y' = x^2 \) given \( y(0) = 0 \) on \([0, 1]\).
6.8 Inverse Trig Fns and Their Derivatives

For graph of \( y = \sin x \), it doesn’t pass the horizontal line test \Rightarrow it doesn’t have an inverse function.

But, if we restrict the domain (to half a period), then we can talk about an inverse.

\[
\begin{align*}
\text{Defn} & \\
\theta = \sin^{-1} y & \iff y = \sin \theta & \quad \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \\
\theta = \cos^{-1} y & \iff y = \cos \theta & \quad \theta \in \left[ 0, \pi \right] \\
\text{Notation} & \\
\arccos(y) &= \cos^{-1}(y) \quad \left( \neq \frac{1}{\cos x} \right)
\end{align*}
\]

Example: Evaluate (w/o calculator).

(a) \( \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \)

(b) \( \sin^{-1} (1) \)

(c) \( \cos^{-1} (\cos (\frac{7\pi}{4})) \)

(d) \( \sin^{-1} (\sin (\frac{3\pi}{2})) \)
6.8 (cont.)

\[ y = \tan x \]

\text{restrict domain to } x \in (-\frac{\pi}{2}, \frac{\pi}{2})

\text{inverse for}

\[ y = \sec x \]

\text{restrict domain to } x \in [0, \pi], x \neq \frac{\pi}{2}

**Defn**

\[
x = \tan^{-1} y \Leftrightarrow y = \tan x, x \in (-\frac{\pi}{2}, \frac{\pi}{2})
\]

\[
x = \sec^{-1} y \Leftrightarrow y = \sec x, x \in [0, \pi] \cup (\pi, \frac{3\pi}{2})
\]

**Ex 2** Evaluate.

(a) \( \tan^{-1} (-1) \)

(b) \( \sec^{-1} (2) \)

\[
\text{Note } \cos \theta = \frac{a}{h} \text{ and } \sec \theta = \frac{h}{a}
\]

\[
\Rightarrow \theta = \cos^{-1} \left( \frac{a}{h} \right) + \theta = \sec^{-1} \left( \frac{h}{a} \right)
\]

\[
= \cos^{-1} \left( \frac{a}{h} \right) = \sec^{-1} \left( \frac{h}{a} \right)
\]

\[
= \sec^{-1} \left( \frac{h}{a} \right) = \cos^{-1} \left( \frac{1}{\frac{h}{a}} \right)
\]

\[
\Rightarrow \quad \sec^{-1} (y) = \cos^{-1} \left( \frac{1}{y} \right)
\]
6.8 (cont)

Let \( \theta = \cos^{-1} x \). Then the rt. \( \Delta \) shown here depicts that info.

What is \( \sin(\cos^{-1} x) \)?

\[
\sin(\cos^{-1} x) = \sin \theta = \frac{d}{1} = d
\]

\[
\Rightarrow x^2 + d^2 = 1 \quad \Rightarrow \quad d^2 = 1 - x^2 \quad \Rightarrow \quad d = \sqrt{1-x^2}
\]

\[
\Rightarrow \sin(\cos^{-1} x) = \sqrt{1-x^2}
\]

(why does this have to be positive?)

Also, \( \sin x = x \Rightarrow x = \sin^{-1} x \)

\[
\cos(\sin^{-1} x) = \cos x = d = \sqrt{1-x^2}
\]

i.e., \( \cos(\sin^{-1} x) = \sqrt{1-x^2} \)

\[
\tan \theta = x \Rightarrow \theta = \tan^{-1} x
\]

\[
\sec(\tan^{-1} x) = \sec \theta = \frac{1}{\cos \theta} = \frac{1}{\sqrt{1+x^2}}
\]

\[
\tan(\sec^{-1} x) = ?
\]

let \( \theta = \sec^{-1} x \)

\[
\Rightarrow \tan(\sec^{-1} x) = \tan \theta = \sqrt{1-x^2}
\]

but we have to consider sign on this one.

\[
\tan(\sec^{-1} x) = \begin{cases} \sqrt{1-x^2} & x \geq 1 \\ -\sqrt{1-x^2} & x \leq -1 \end{cases}
\]

if \( x \geq 1 \), in QI

if \( x \leq -1 \), in QII
6.4 (cont.)

\[\begin{align*}
D_x (\sin x) &= \cos x \\
D_x (\tan x) &= \sec^2 x \\
D_x (\sec x) &= \sec x \tan x \\
D_x (\cot x) &= -\csc^2 x \\
D_x (\csc x) &= -\csc x \cot x
\end{align*}\]

Ex 3 Calculate \[\sin \left[ 2 \cos^{-1} \left( \frac{1}{4} \right) \right]\] (w/o calculator).

Derivatives of Inverse Trig. Fns

Let \( y = \cos^{-1} x \). We want to find \( y' \).

\( \equiv \) \( x = \cos y \)

\[D_x [x] = D_x [\cos y] \]

\[1 = (-\sin y) y' \]

\[y' = \frac{-1}{\sin y} = \frac{-1}{\sin (\cos^{-1} x)} = \frac{-1}{\sqrt{1-x^2}}\]

\[\begin{align*}
D_x (\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}} \quad x \in (-1, 1) \\
D_x (\sec^{-1} x) &= \frac{1}{|x| \sqrt{x^2-1}} \quad |x| > 1 \\
D_x (\cos^{-1} x) &= \frac{-1}{\sqrt{1-x^2}} \quad x \in (-1, 1) \\
D_x (\tan^{-1} x) &= \frac{1}{1+x^2}
\end{align*}\]
\[ \int \frac{1}{1-x^2} \, dx = \sin^{-1} x + C \quad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C \]

\[ \int \frac{1}{x \sqrt{x^2 - 1}} \, dx = \sec^{-1} |x| + C \]

Ex. 4 \[ D_x \left[ \tan^{-1} (5x^2 - 3x + 1) \right] = ? \]

Ex. 5 Evaluate integrals

(a) \[ \int_{-1}^{1} \frac{1}{1+x^2} \, dx \]

(b) \[ \int \frac{e^x}{1+e^{2x}} \, dx \]
6.9 Hyperbolic Fns. & Their Inverses

Hyperbolic Functions

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} \\
\sech x &= \frac{1}{\cosh x} \\
\csch x &= \frac{1}{\sinh x}
\end{align*}
\]

*Related to trig fns:*

1. \((\cos \theta, \sin \theta)\) pt on unit circle
2. \(\sin \theta\) odd on \(\mathbb{R}\)
3. \(\cos \theta\) even on \(\mathbb{R}\)
4. \(\sin^2 \theta + \cos^2 \theta = 1\)

Prove \(\cosh^2 \theta - \sinh^2 \theta = 1\)

\[\text{pf}\]

\[\text{Math1220}\]

\[\text{38}\]
6.9 (cont)

\[ D_x (\cosh x) = D_x \left( \frac{e^x + e^{-x}}{2} \right) = \frac{1}{2} (e^x - e^{-x}) = \sinh x \]

\[ D_x (\sinh x) = \cosh x \]

\[ D_x (\tanh x) = \text{sech}^2 x \]

\[ D_x (\text{sech} x) = -\text{sech} x \tanh x \]

\[ D_x (\cosh x) = \sinh x \]

\[ D_x (\coth x) = -\text{csch}^2 x \]

\[ D_x (\text{csch} x) = -\text{csch} x \cot x \]

**Ex 1**

\[ D_x (\coth (4x) \sinh x) = ? \]

**Ex 2**

\[ \int \tanh x \ln (\cosh x) \, dx \]
Ex 3 Verify identity.

\[
\tanh (x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}
\]

(Hint: \( \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \))
6.9 (cont)

Inverse Hyperbolic Fns

Let \( y = \sinh x \Rightarrow x = \sinh^{-1} y \) (if inverse exists).

Find \( x = \sinh^{-1} y \).

\[
y = \sinh x = \frac{e^x - e^{-x}}{2}
\]

\[
2y = e^x - e^{-x}
\]

\[
e^x \left( 2y \right) = \left( e^x - \frac{1}{e^x} \right) e^x
\]
\[ \sinh^{-1} x = \ln (x + \sqrt{x^2 + 1}) \quad \cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}) \quad x \geq 1 \]
\[ \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) \quad x \in (-1, 1) \quad \text{sech}^{-1} x = \ln \left( \frac{1 + \sqrt{1 - x^2}}{x} \right) \quad x \in (0, 1] \]

\[ D_x (\sinh^{-1} x) = D_x \left( \ln (x + \sqrt{x^2 + 1}) \right) \]
\[ = \frac{1}{x + \sqrt{x^2 + 1}} \left( 1 + \frac{2x}{2\sqrt{x^2 + 1}} \right) \]
\[ = \frac{1}{x + \sqrt{x^2 + 1}} \left( \sqrt{x^2 + 1} + x \right) \]
\[ = \frac{1}{\sqrt{x^2 + 1}} \]

\[ D_x (\sinh^{-1} x) = \frac{1}{\sqrt{x^2 + 1}} \quad D_x (\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}} \quad x > 1 \]

\[ D_x (\tanh^{-1} x) = \frac{1}{1 - x^2} \quad x \in (-1, 1) \quad D_x (\text{sech}^{-1} x) = \frac{-1}{x \sqrt{1 - x^2}} \quad x \in (0, 1) \]

**Ex 4**
Find \( y' \):
\[ y = x^2 \sinh^{-1} (x^5) \]
**Log Properties**

\[
\ln 1 = 0
\]

\[
\ln(ab) = \ln a + \ln b
\]

\[
\ln\left(\frac{a}{b}\right) = \ln a - \ln b
\]

\[
\ln(a^m) = m \ln a
\]

**Integrals**

\[
\int \frac{1}{x} \, dx = \ln |x| + C
\]

\[
\int e^x \, dx = e^x + C
\]

\[
\int \frac{a^x}{\ln a} \, dx = \frac{a^x}{\ln a} + C
\]

\[
\int x^a \, dx = \frac{x^{a+1}}{a+1} + C
\]

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1}(x) + C
\]

\[
\int \frac{1}{1+x^2} \, dx = \tan^{-1}(x) + C
\]

\[
\int \frac{2}{x^2 - 1} \, dx = \sec^{-1} |x| + C
\]

**Functions and Identities**

\[
\sin(\cos^{-1} x) = \sqrt{1-x^2}
\]

\[
\cos(\sin^{-1} x) = \sqrt{1-x^2}
\]

\[
\sec(\tan^{-1} x) = \sqrt{1+x^2}
\]

\[
\tan(\sec^{-1} x) = \sqrt{x^2-1}, \text{ if } x > 1
\]

\[
-\sqrt{x^2-1}, \text{ if } x \leq -1
\]

\[
\sin^{-1} x = \ln(x + \sqrt{x^2+1})
\]

\[
\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), x \geq 1
\]

\[
\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), -1 < x < 1
\]

\[
\sec h^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), 0 < x \leq 1
\]

\[
\sinh x = \frac{e^x - e^{-x}}{2}
\]

\[
\cosh x = \frac{e^x + e^{-x}}{2}
\]

**Trig Identities**

\[
\sin^2 x + \cos^2 x = 1
\]

\[
1 + \tan^2 x = \sec^2 x
\]

\[
1 + \cot^2 x = \csc^2 x
\]

\[
\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y
\]

\[
\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y
\]

\[
\frac{\tan x \pm \tan y}{1 + \tan x \tan y}
\]

\[
\sin 2x = 2 \sin x \cos x
\]

\[
\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1
\]
### Derivatives

<table>
<thead>
<tr>
<th>$D_x(e^x)$</th>
<th>$e^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_x(\ln x)$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$D_x(x^n)$</td>
<td>$ax^{n-1}$</td>
</tr>
<tr>
<td>$D_x(a^x)$</td>
<td>$(\ln a)a^x$</td>
</tr>
<tr>
<td>$D_x(\sin x)$</td>
<td>$\cos x$</td>
</tr>
<tr>
<td>$D_x(\cos x)$</td>
<td>$-\sin x$</td>
</tr>
<tr>
<td>$D_x(\tan x)$</td>
<td>$\sec^2 x$</td>
</tr>
<tr>
<td>$D_x(\cot x)$</td>
<td>$-\csc^2 x$</td>
</tr>
<tr>
<td>$D_x(\sec x)$</td>
<td>$\sec x \tan x$</td>
</tr>
<tr>
<td>$D_x(\csc x)$</td>
<td>$-\csc x \cot x$</td>
</tr>
<tr>
<td>$D_x(\sin^{-1} x)$</td>
<td>$\frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$</td>
</tr>
<tr>
<td>$D_x(\cos^{-1} x)$</td>
<td>$\frac{-1}{\sqrt{1-x^2}}$, $x \in (-1,1)$</td>
</tr>
<tr>
<td>$D_x(\tan^{-1} x)$</td>
<td>$\frac{1}{1+x^2}$</td>
</tr>
<tr>
<td>$D_x(\sec^{-1} x)$</td>
<td>$\frac{1}{</td>
</tr>
</tbody>
</table>

### More Derivatives

| $D_x(\sinh x)$ | $\cosh x$ |
| $D_x(\cosh x)$ | $\sinh x$ |
| $D_x(\csc x \cdot \cot x)$ | $-\csc x \cdot \coth x$ |
| $D_x(\tanh x)$ | $\text{sech}^2 x$ |
| $D_x(\sec x)$ | $-\sec x \cdot \tanh x$ |
| $D_x(\coth x)$ | $-\csc^2 x$ |
| $D_x(\sinh^{-1} x)$ | $\frac{1}{\sqrt{x^2+1}}$ |
| $D_x(\cosh^{-1} x)$ | $\frac{1}{\sqrt{x^2-1}}$, $x > 1$ |
| $D_x(\tanh^{-1} x)$ | $\frac{1}{1-x^2}$, $-1 < x < 1$ |
| $D_x(\sec^{-1} x)$ | $\frac{-1}{x\sqrt{1-x^2}}$, $0 < x < 1$ |

### Miscellaneous

$$\lim_{h \to 0}(1+h)^\frac{1}{h} = e$$

To solve $\frac{dy}{dx} + P(x)y = Q(x)$, multiply both sides by $e^{\int P(x)dx}$ and solve.